Topic 1 Sequences & Proof

1

The Sigma Notation

$\sum_{r=0}^{5} 2r + 1$ means that

$(2 \times 0 + 1) + (2 \times 1 + 1) + ... + (2 \times 5 + 1)$

Where the top number of the sigma notation denotes how many times r will be substituted in a pattern of $r, r + 1, r + 2$ etc.

The bottom number denotes the first number of r that will be substituted for.

2

Arithmetic Sequences and Series

An arithmetic sequence holds a general pattern when counting the next term which is usually easily detectable.The common difference in an arithmetic sequence is denoted as - d.

For example: 2,5,8,11.. Etc. is an arithmetic sequence.

One way to find the arithmetic sequence is to first look at the difference between each term. The difference is 3, hence d can be expressed as $3d$. Substituting into the first term, we get that it is minused by one, hence

$U_r = 3d - 1$

An expression for an arithmetic sequence general term can be given as the following:

$U_n = u_1 + (n - 1)d$ (SL 1.2)

Where
u_1 is the first term
n is the term's order number
U_n is the term's order number resultant
d is the common difference

The equation of an arithmetic sequence is called a **series.**

It is given using 2 formulas:

$S_n = \frac{n}{2}(2u_1 + (n - 1)d$ or $S_n = \frac{n}{2}(u_1 + u_n)$(SL 1.2)

Where
u_1 is the first term
u_n is the last term
n is the number of terms
d is the common difference

Otherwise, the sum of an arithmetic series can be shown symbol is shown below:

Example 12

Find the value of $\sum_{r=1}^{28} 5r - 4$.

$u_1 = 1$ $u_{28} = 140 - 4 = 136$ $S_{28} = \frac{28}{2}(1 + 136) = 1918$	Substitute $r = 1$ and $r = 28$ to find the first and last terms. Using the formula $S_n = \frac{n}{2}[u_1 + u_n]$.

Moreover,
The series sequence is like this:
$S_n = n,$

3

Geometric Sequences

Geometric sequences are similar to arithmetic sequences, however, instead of adding a common difference d, we multiply by common ratio r.

For example,
1, 2, 4, 8, 16, 32, 64....

Is a geometric sequence with common ratio $r = 2$. Common ratio r can be found by dividing a higher term by one lower term.

For example, a common ratio can be expressed as

$u_1, ru_1, r^2u_1, r^3u_1, r^4u_1, r^5u_1$ etc.

And if we divide, for example a higher term by one lower term we get,

$$\frac{r^5u_1}{r^4u_1} = r$$

The nth term of a geometric series can be given by

$$u_n = u_1r^{n-1} \text{(SL 1.3)}$$

Where
u_n is the term's order number resultant
u_1 is the first term
r is the common ratio

The sum of a geometric series can be given by

$$S_n = \frac{u_1(r^n-1)}{r-1} \text{ or } \frac{u_1(1-r^n)}{1-r} \quad r \neq 1 \text{(SL 1.3)}$$

Where
u_1 is the first term

$r$is the common ratio

The sum of an infinite geometric series can be given by

$S_{\infty} = \frac{u_1}{1-r},\ |r| < 1$thus $-\ 1 < r < 1$(SL 1.8)

If a geometric series switch positive, negative, positive negative then it is possible that $(-\ 1)^n$is a part of the equation.
In order to find the convergence inequality in a sum of infinite geometric series, substitute r into $-\ 1 < r < 1$.

4

Interest and Geometric Series

The following example is found in page 30, example 27 in the book:

When Jacob turned 18 he had access to the money his grandparents had invested in a savings account. He decided to re-invest $10,000 at a compound rate of 3% each year. He decided to add $200 to this investment on his next birthday and each following birthday until he turned 25. Evaluate how much was in his account just after his 25th birthday. Evaluate the total interest gained over this time.

$= (((10000 \times 1.03 + 200) \times 1.03 + 200) \times 1.03 + 200)\ldots\ etc.$

Hence, we realise that that $10000$is multiplied by 1.037times, or **otherwise** we times the left side by 1.03 each time as expressed in the equation

10000×1.03^{7}

We also realise that each investment that is worth $200$is also multiplied by 1.03 depending on the time of the initial investment. The first 200investment will be multiplied by 1.03 6times because it went through 6 other years/investments.

Thus:

$= 10000 \times 1.03^{7} + (200 \times 1.03^{6} + 200 \times 1.03^{5} + 200 \times 1.03^{4} + 200 \times 1.03^{3}$
$+ 200 \times 1.03 + 200)$

$= 10000 \times 1.03^{7} + 200(1.03^{6} + 1.03^{5} + 1.03^{4} + 1.03^{3} + 1.03^{2} + 1.03 + 1)$

We see a geometric sequence forming on the right side with u_1= 1, $r = 1.03$and $n = 7$thus we use the sum of geometric series formula

Hence:

$$= 10000 \times 1.03^{7} + 200\left(\frac{1(1.03^{7}-1)}{1.03-1}\right)$$
$$= \$13831.23$$

∴ he gained $13831.23 - (10000 + 7 \times 200) = \$2431.23$

5

Direct Proof

Direct proof is when a proof of a statement is done through showing constructing a series of reasoned connected established facts. In a direct proof the following steps are used:

- Identify the given statement
- Use axioms, theorems, etc, to make deductions to prove the conclusion of your statement is true

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It is essential to know these:

When a number is defined **odd**, it is defined as 2n-1
When a number is defined **even**, it is defined as 2n.
9 and 3 divisibility rule:
Let a four digit number be defined by
Abcd
If $a + b + c + d = 3k$
Then $abcd = 3z$
Where you can find z in terms of $3k$ and $abcd$ (not shown in the example) (If attempting to solve remember that since $a$is the front term, then $a = 3x \times 10^3$)
This also applies for the number 9.

For example (Example 28, Page 37):

Show that
$1 + 3 + 5 + 7... + (2n - 1) = n^2$
We write that, including extra reversed terms
$S = 1 + 3 + 5 + 7... + (2n - 5) + (2n - 3) + (2n - 1)$
We write it in the reverse order
$S = (2n - 1) + (2n - 3) + (2n - 5) +... + 7 + 5 + 3 + 1$
We sum the two
$2S = 2n + 2n + 2n + 2n....$
The sum of $2n$ happens $n$times (meaning there are n of $2n$), thus
$2S = 2n^2$
$S = n^2$

Another example (Example 29, Page 38):

Show that:

a. The sum of an odd and even positive integer is always odd
b. The sum of two even numbers is always even
c. The sum of two odd numbers is always even

a.
Let us denote that
Odd number is expressed by $a = 2z - 1$
Even number is expressed by $b = 2y$
Hence
$a + b = 2z - 1 + 2y$
$a + b = 2(z + y) - 1$

b.
$a = 2z$
$b = 2y$
Hence
$a + b = 2z + 2y$
$a + b = 2(z + y)$

c.
$a = 2z - 1$
$b = 2y - 1$
Hence
$a + b = 2z - 1 + 2y - 1$
$a + b = 2(z + y - 1)$

(These examples don't have conclusions and don't show the elements of the unknown letters, make sure to include them!)

Another example (example 33, page 39):

Show that

$$\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \frac{1}{32} - \frac{1}{64} + \ldots = \frac{1}{3}$$

We split the equation into 2 different geometric series:

LHS

$$\left(\frac{1}{2}+\frac{1}{8}+\frac{1}{32}\ldots\right)-\left(\frac{1}{4}+\frac{1}{16}+\frac{1}{64}\ldots\right)$$

We notice that both are converging infinite geometric series of whos $|r|<1$, hence we subtract sums

$$\frac{\frac{1}{2}}{1-\frac{1}{4}}-\frac{\frac{1}{4}}{1-\frac{1}{4}}=\frac{2}{3}-\frac{1}{3}=\frac{1}{3}$$

6

Proof by Contradiction

Proof by contradiction is when you create an opposite of a statement and prove that the opposite statement is true or false.

It is essential to know these:

When a number is defined **odd**, it is defined as 2n-1
When a number is defined **even**, it is defined as 2n.
9 and 3 divisibility rule:
Let a four digit number be defined by
Abcd
If
$a + b + c + d = 3k$
Then $abcd = 3z$
Where you can find z in terms of $3k$ and $abcd$ (not shown in the example) (If attempting to solve remember that since $a$is the front term, then $a = q \times 10^3$)
This also applies for the number 9.

Example (Example 34, Page 42):

Prove by contradiction:

a. If n is odd then $n^2$is also odd
b. If $n^2$is even then n is also even

a. If n is odd then $n^2$is even

$n^2 = 2k$
$n \times n = 2k$

Statement can not be true because if n is odd then the product of 2 n is also odd.

b. If $n^2$is even then n is odd

$n = 2k - 1$
$n^2 = (2k - 1)^2$
$n^2 = 4k^2 - 4k + 1$
$n^2 = 2(2k^2 - 2k) + 1$

$n^2 = 2p + 1$
Hence it's odd, thus if odd makes odd then even makes even.

Example (Example 35, Page 42):

Show that $\sqrt{2}$ is irrational.

$\sqrt{2} = \frac{m}{n}$
$2 = \frac{m^2}{n^2}$
$2n^2 = m^2$
Which means that m is an even integer

Let $m = 2k$ hence $m^2 = 4k^2 = 2n^2$

$2n^2 = 4k^2$

Both have a factor of 2, which is not prime. Recall we have assumed that they $\sqrt{2}$ can be expressed as prime numbers m, n. They have a common factor. As such, you cannot find $\frac{m}{n}$with no common factors.

Example (Example 37, Page 43):

Prove that if $m,\ n \in Z$,then $m^2 - 4n - 7 \neq 0$

$m^2 - 4n - 7 = 0$
$m^2 = 4n + 7$
$m^2 = 2(2n + 3) + 1$
$m^2$is hence odd
Let $m = 2p - 1$
$(2p - 1)^2 - 4n - 7 = 0$
$4p^2 - 4p + 1 - 4n - 7 = 0$
$2p^2 - 2p - 2n - 3 = 0$
$2(p^2 - p - n) = 3$
Cannot be true because it is even since it is multiplied by 2.

7

Proof by Counterexample

Proof by counterexample is when a statement is proven wrong by showing an example which contradicts the statement.

Example (Example 38, Page 44):

If $n \in Z$and $n^2$is divisible 4, then n is divisible by 4

$n = 2,\ n^2 = 4$

The squared is divisible by 4, however, n is not.

8

Proof by Induction

Proof by induction is by proving for one number, and a number higher than that. It is like a domino, if you hit one domino it hits the others. As such, if you prove that the one higher is true, that means higher and higher will also be true.

We follow the following steps:

B.A.I.C

Base step: where you substitute a number for n, and prove that the statement is true

Assumption: you substitute n for k

Inductive step: you substitute n for $k + 1$ and do algebraic manipulation. What happens here is usually the following:

… an addition sequence with k = sum equation of sequence with k (as defined in the assumption step)
…. An addition sequence with $k + 1$ = sum equation of sequence with $k + 1$

To prove, you do

Sum equation of sequence with k + the **difference** of addition sequence with $k + 1$ term = sum equation of sequence with k+1

This happens because the $k + 1$ term has all the terms that k has, but an extra k+1 term.

Note for the **difference:** alternatively and almost most likely, you will substitute the sum equation of sequence with k into addition sequence with $k + 1$term.

For example:

$1 + 2 + 3 + \ldots + (k-1) + k + (k-1) + \ldots + 3 + 2 + 1 = k^2$

When $n = k + 1$ LHS

$= 1 + 2 + 3 + \ldots + (k-1) + k + (k+1) + k + (k-1) + \ldots + 3 + 2 + 1$

$= 1 + 2 + 3 + \ldots + (k-1) + k + (k-1) + \ldots + 3 + 2 + 1 + (k+1) + k$

$= k^2 + (k+1) + k$

$= k^2 + 2k + 1$

$= (k+1)^2$

Example 39 page 46

ie $11^k - 6 = 5b$, where $n, b \in \mathbb{Z}^+$

$\Rightarrow 11^k = 5b + 6$

When $n = k + 1$:

LHS

$= 11^{k+1} - 6$

$= 11 \times 11^k - 6$

$= 11(5b + 6) - 6$

$= 55b + 66 - 6$

$= 55b - 60$

$= 5(11b - 12)$

Example 40 page 47

Conclusion: Since base step for X is true, and when assumed true for some $n = k$ where $k \in Z^+$(depending on question), it is also true for $n = k + 1$. Then by the principle of mathematical induction that the statement is true for all positive integers.

9

Permutations & Combinations

Combinations are used when there are no rules to the combinations of something.

Permutations are used when there is a set of rules for the combinations of something.

The formula for combinations is the following:

$$\binom{n}{r} \text{or } {}^{n}C_{r} = \frac{n!}{r!(n-r)!} \text{ (AHL 1.10)}$$

The formula for permutations is the following:

$${}^{n}P_{r} = \frac{n!}{(n-r)!} \text{ (AHL 1.10)}$$

Where
n is the number of distinct objects
r is the number of objects

General rules that must be known for different type of questions:

When something is treated as a group, such as books in a bookshelf all in the history category, treat it as a single object.

You can find probability by putting the desired combination over all possible combinations.

If a multi object is treated as a single object, remember that this single object can have combinations inside it too. (Example 1)

Permutations don't have to be expressed in the formula, and can be done so logically. (Example 3)

Sometimes it is easier to subtract combinations from the total combinations in order to get an answer. (Example 2)

Identity:

$${}^{n}C_{r} = {}^{n}C_{n-r} \text{ because of symmetry}$$

$0! = 1$, either because there is only a single way to arrange nothing, or $\frac{(n+1)!}{(n+1)} = \frac{1}{1}$ when $n = 0$

Example 1 (Exercise 1H, 7, Page 57):

On a bookshelf there are four mathematics books, three science books, two geography books and four history books. The books are all different.

a. In how many different ways can the book be arranged on the shelf?

b. In how many ways can the books be arranged so that the books of the same subject can be grouped together?

a.
The books can be arranged $13!$ ways, because there is nothing specified.

b.
We group each subject and arrange in the most ways possible in their own groups. For example, there are 4 history books, hence there are $4!$ Ways of arranging history books when grouped. We do this for every subject and multiply them together

$4! \times 3! \times 2! \times 4!$

However, we also need to realise that there are $4!$ways of arranging the book groups together, hence

$4! \times 3! \times 2! \times 4! \times 4! = 165888$ ways of arranging

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Example 2 (Example 47, Page 56):

There are 8 boys and 5 girls that attend the senior mathematics club. Find out how many ways a teacher can choose a team of 6 students to represent the school in a competition if:

a. There are no gender restrictions

b. The team is to be made of 3 girls and 3 boys

c. At least two of each gender are included in the team

a.
Since we are choosing 6 candidates from a total of 13:

$^{13}C_6 = \frac{13!}{6!(13-6)!} = 1716$

b.
We split the group of boys and girls up, and choose from them. Hence:

$^{8}C_3 \times \ ^{6}C_3 = \frac{8!}{3!(8-3)!} \times \frac{6!}{3!(6-3)!} = 1120$

c.
We will subtract on what is not possible from the total.

It is impossible to have 6 boys, hence
$^{8}C_6 \times \ ^{5}C_0$

It is impossible to have 5 boys and 1 girl, hence
$^{8}C_5 \times \ ^{5}C_1$

It is impossible to have 1 boy and 5 girls, hence
$^{8}C_1 \times \ ^{5}C_5$

Subtracting from the total we get
$^{13}C_6 - \ ^{8}C_1 \times \ ^{5}C_5 - \ ^{8}C_5 \times \ ^{5}C_1 - \ ^{8}C_6 \times \ ^{5}C_0 = 1400$

Example 3 (Exercise 1H, 10, Page 57):

a. How many four digit numbers can be made using the digits $0, 1, 3, 4, 5, 8, 9$?

b. How many four digits numbers have no repeated digits?

c. How many four digit even numbers can be made using the digits?

d. How many four digit even numbers can be made that are divisible by5?

a.
We need to notice that we can not have the 0 in the beginning, otherwise it would be a 3 digit. We choose the amount of possibilities for each digit. For example, we can only have 6 digits in the beginning because we exclude 0.

$6 \times 7 \times 7 \times 7 = 2058$

b.
Similar concept follows, however, the amount of numbers we can choose will lessen gradually as we use them up in creating our digit:

$6 \times 6 \times 5 \times 4 = 720$

c.
For it to be even, we have to make sure that it ends in $0,\ 4$ or 8.

$6 \times 7 \times 7 \times 3 = 882$

d.
For it to be even and also be divisible by 5, we have to make that it ends in 0.

$6 \times 7 \times 7 \times 1 = 294$

Example 4 (made-up):

In how many different ways can the letters in the word ***calculator*** be arranged?

We notice that it has repeating letters a, c and l.
Thus we put them in a group. There are 2 of each. We have to have a total of 10 letters to choose. We divide the repeating groups in factorial notation over the total possibilities. Hence:

$\frac{10!}{2! \times 2! \times 2!} = 453600$

10

Binomial Theorem

The binomial expansion helps expanding brackets and finding the coefficients in big expansions.

The binomial theorem for when $x \in R$, $x \geq 1$, $n \geq 1$

$(a + b)^n = a^n + {}^nC_1 a^{n-1} b + ... + {}^nC_r a^{n-r} b^r + ... + b^n$(SL 1.9)

The binomial theorem for when $n \in R$ and $a = 1, -1$ (with -1 the signs would be alternating)

$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + ... + \frac{n(n-1)(n-2)...(n-r+1)}{r!}x$
(NOT GIVEN IN THE FORMULA BOOKLET!)

Proof:

Let $(1 + x)^n$ be a binomial expansion. Then:
$1^n + {}^nC_1 \times 1^{n-1} \times x^1 + {}^nC_2 \times 1^{n-2} \times x^2 + {}^nC_3 \times 1^{n-3} \times x^3 + ... + x^n$
The 1s cancel out. Let us express the Cin terms of factorials, then:
$1 + \frac{n!}{1!(n-1)!} \times x + \frac{n!}{2!(n-2)!} \times x^2 + \frac{n!}{3!(n-3)!} \times x^3 + ... + x^n$
Cancelling out the factorials we get
$1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + ... + x^n$
QED

This formula should also be used when the power of an expansion is unusual looking. It is called Newton's generalisation of the binomial theorem.

I.e $(1 + 2x)^{\frac{3}{4}}$
If a term has no 1 inside with an unusual power, then look to factor out. E.g

$(5 + 10x)^{-\frac{1}{2}}$

$((1 + 2x) \times 5)^{-\frac{1}{2}}$

$(1 + 2x)^{-\frac{1}{2}} \times 5^{-\frac{1}{2}}$ and expand the bracket on left using the above formula.

When a question asks about x^n in an expansion, you equate the x^n to the general binomial term of ${}^{n}C_{r}\, a^{n-r} b^{r}$ (Examples 1, 3)

When it asks to estimate, you will have to give proper suitable value for x and y. (Example 2)

When it asks for the independent term, you equate x^0 to the general binomial term of ${}^{n}C_{r}\, a^{n-r} b^{r}$, or otherwise you find the coefficient that has no x in its expansion.

When 2 binomials expansions are multiplied together, it follows this form (Example 5):

let f(x)= 1+a+b+c+d+e...
let g(x)= 1+z+y+x+w+v...

When producted:

1	+ z	+ y	+ x	+ w	+ v
	+ a	+ az	+ ay	+ ax	+ aw
		+ b	+ bz	+ by	+ bx
			+ c	+ cz	+ cy
				+ d	+ dz
					+ e

Example 1 (Example 49, Page 60):

Find the coefficient x^3y^3 in the expansion of $(x + 3y)^6$.

The general term in this expansion is given as

$$ {}^{6}C_{r}\, x^{6-r}(3y)^{r} = x^{3}y^{3} $$

We equate the powers to find r. For example, in x component we get that

$x^{6-r} = x^{3}$
$r = 3$

To find the coefficient, we have calculate all the numbers which are not stuck to a letter. For example, $3y$. The 3 is a coefficient thus we have to include it.

${}^{6}C_{3} \times 3^{3} = 540$is the coefficient.

Example 2 (Example 50, Page 60):

Use the binomial theorem to expand $(2x + 3y)^{5}$. Hence find the value of 2.03^{5} correct to 5 decimal places.

We expand using the binomial theorem:
$= (2x)^{5} + {}^{5}C_{1}(2x)^{4}(3y) + {}^{5}C_{2}(2x)^{3}(3y)^{2} + {}^{5}C_{3}(2x)^{2}(3y)^{3} + {}^{5}C_{4}(2x)(3y)^{4} + (5$

We notice that $2x + 3y = 2.03$, hence we can deduce that $x = 1$and $y = 0.01$

=$32x^{5} + 240x^{4}y + 720x^{3}y^{2} + 1080x^{2}y^{3} + 810xy^{4} + 243y$
We substitute and get
$= 32.47309$

Example 3 (Exam-style questions, 17, Page 69):

Find the coefficient of the term in x^{5}in the binomial expansion of $(3 + x)(4 - 2x)^{8}$

Using FOIL, we can get that

$3(4 - 2x)^{6} + x(4 - 2x)^{6}$

Working with the left bracket first we do the following
$3({}^{8}C_{r}4^{8-r}(- 2x)^{r}) = x^{5}$
$r = 5$
As such, we get the coefficient as
$3({}^{8}C_{5}4^{3}(- 2)^{5}) =- 344064$

Working with the right bracket we do the following
$x({}^{8}C_{r}4^{8-r}(- 2x)^{r}) = x^{5}$
$x({}^{8}C_{r}4^{8-r} - 2^{r}x^{r}) = x^{5}$
${}^{8}C_{r}4^{8-r} - 2^{r}x^{r+1} = x^{5}$
$r = 4$
As such, we get the coefficient as
${}^{8}C_{4}4^{4}(- 2)^{4} = 286720$

Hence we get
$-\ 344064\ +\ 286720\ =-\ 57344$as the coefficient

Example 4 (Exercise 7I, 6 b, Page 61):

Find the coefficient of x^3y^2in the expansion of $(2x\ +\ y)(x\ +\ \frac{x}{y})^5$

$^5C_r(x)^{5-r}(\frac{y}{x})^r = x^3y^2$

$^5C_r x^{5-r}x^{-r}y^r = x^3y^2$

$^5C_r x^{5-2r}y^r = (x^3y)y$

$r\ =\ 1$hence coefficient is 5.

Example 5 (Example 53, Page 64):

Use the binomial expansion to show that $\sqrt{\frac{1+x}{1-x}} \simeq\ 1\ +\ x\ +\ \frac{1}{x^2}, |x|<\ 1$

We rewrite it as

$(1\ +\ x)^{\frac{1}{2}}(1\ -\ x)^{-\frac{1}{2}}$

We expand each bracket independently:

$(1\ +\ x)^{\frac{1}{2}}\ =\ 1\ +\ 0.5x\ +\ \frac{0.5(0.5-1)}{2!}x^2\ +\ \frac{0.5(0.5-1)(0.5-2)}{3!}x^3$

$=\ 1\ +\frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16}$

$(1\ -\ x)^{-\frac{1}{2}}\ =\ 1\ +\ (-\ 0.5)(-\ x)\ +\ \frac{-0.5(-0.5-1)}{2!}(-\ x)^2\ +\ \frac{-0.5(-0.5-1)(-0.5-}{3!}$

$=\ 1\ +\frac{x}{2} + \frac{3x^2}{8} + \frac{5x^3}{16}$

Since it only asks for the first 3 terms in the question, we start with multiplying the first term with the terms in the right equation. As we shift terms in the left side, we do so in the answer as well. As such, for example, the second term will begin as a second term and will be defined as the product of the 2nd term in left with 1st term in right.

$= 1 + \frac{x}{2} + \frac{3x^2}{8}$(for example, this is the first term multiplied to the right side)

$+ \frac{x}{2} + \frac{x^2}{4}$ (for example, this is $\frac{x}{2}$multiplied as the second term to the right side)

$- \frac{x^2}{8}$(for example, this is 1 multiplied by the 3rd term from the left side)

$\simeq 1 + x + \frac{x^2}{2}$

Example 6 (Exercise 7J, 6, Page 65):

a. Find the first four terms of the binomial expansion of $\sqrt{1 - 4x},\ |x| < \frac{1}{4}$

b. Show that the exact value of $\sqrt{1 - 4x}$ when $x = \frac{1}{100}$is $\frac{2\sqrt{6}}{5}$.

c. Hence, determine the value of $\sqrt{6}$to 5 decimal places.

a.
We first rewrite it as the following

$(1 - 4x)^{\frac{1}{2}} = 1 - 2x + \frac{0.5(0.5-1)}{2!}(-4x)^2 + \frac{0.5(0.5-1)(0.5-2)}{3!}(-4x)^3$

$= 1 - 2x - 2x^2 - 4x^3$...

b.

$\sqrt{1 - \frac{1}{25}} = \sqrt{\frac{24}{25}} = \sqrt{\frac{2^2\times 6}{5^2}}$

$= \frac{2}{5}\sqrt{6}$

c.
Hence, to first find the value of $\frac{2}{5}\sqrt{6}$ we substitute $x = \frac{1}{100}$into the binomial expansion as it is defined:

$= 1 - \frac{2}{100} - \frac{2}{10000} - \frac{4}{1000000} = \frac{2}{5}\sqrt{6}$

Thus we have to multiply by $\frac{5}{2}$on both sides

$= \frac{5}{2}(1 - \frac{2}{100} - \frac{2}{10000} - \frac{4}{1000000}) = \sqrt{6}$

≈ 2.44949

Topic 2 Functions

1

Domain, Range and Functions' basic properties

The domain of a function or of a graph is the territory in which x values exist.

The range of a function or of a graph is the territory in which y values exist.

NOTE: when solving an inequality, if you multiply or divide both sides by a negative, then you swap the inequality.

It is written in the following form:

Domain: $x \mid x \in R, Z, C, Q, N,\ i.e\ x \mid x \in R,\ x \neq 5$
Range: $y \mid y\ \in R, Z, C, Q, N,\ i.e\ y \mid y \in N^{+},\ y \neq 5$

Function is a type of graph that is defined as:
One to one
Many to one
I.e one x value for one y value ($y = mx + b$graphs)
Or many x values for one y value (x^2graphs)

Onto function
Let us define set A as a, b, c, d set Bas $1, 2, 3$
Let us define that $a = 1, 2$
$b = 3$
$c = 2$
$d = 1$
Notice that, set $A \in B$, and consists of all numbers which are linked to each other, as such it is defined as an onto function.

Other relations include but are **not** defined as functions:
One to many ($y^2 = x$graphs)
Many to many ($1 = y^2 + x^2$graphs)

Another way checking whether a graph is a function is through the vertical line test.

Axis of symmetry of a quadratic function can be found by plotting x value for

$-\frac{b}{2a}$ in $y = ax^2 + bx + c$(SL 2.6)

To find the asymptotes and the intersections of a function:

To find the $y$intercept, substitute $x = 0$

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To find the horizontal asymptotes of a function who has an $x$in the denominator, equate the denominator to 0as for example $\frac{1}{0}$is undefined thus an asymptote.

To find the vertical asymptotes of a function, divide the highest $x^n$of the denominator by the same x^n. If $x^n$is not present in the numerator and/or there are lower powers, that is the same as $\frac{0}{x^n} = 0$. If there are higher powers on the numerator than the denominator, then there are no vertical asymptotes.

2

Vertex Form

The general vertex form of a quadratic is given as

$a(x - h)^2 + k$ where $(h, k)$is the coordinate of the vertex.

Example (made-up):

Put the function $2x^2 + 6x + 40$in the vertex form

We first factories the 2

$2(x^2 + 3x) + 40$

We then divide the inside by 2 and put the square outside, removing x. After doing so, we minus the squared part of the number with no x to the outside. It is important to keep note of the factored 2, because we then times the outside number by 2. It is important to note that the number that the number on the outside will ALWAYS be a minus.

$y = 2((x + 1.5)^2 - 2.25) + 40$

$y = 2((x + 1.5)^2 - 4.5 + 40$

$y = 2(x + 1.5)^2 + 35.5$

Hence the vertex is $(- 1.5, 35.5)$

ALTERNATIVE METHOD

We know that x coordinate of the vertex is $-\frac{b}{2a}$

Hence

$x =- \frac{6}{4}$

Substituting the x value in the quadratic we get

$y = 2(\frac{-3}{2})^2 + 6(-\frac{3}{2}) + 40$

$y = 2(\frac{9}{4}) - 9 + 40$

$y = 4.5 - 9 + 40$

$y = 35.5$

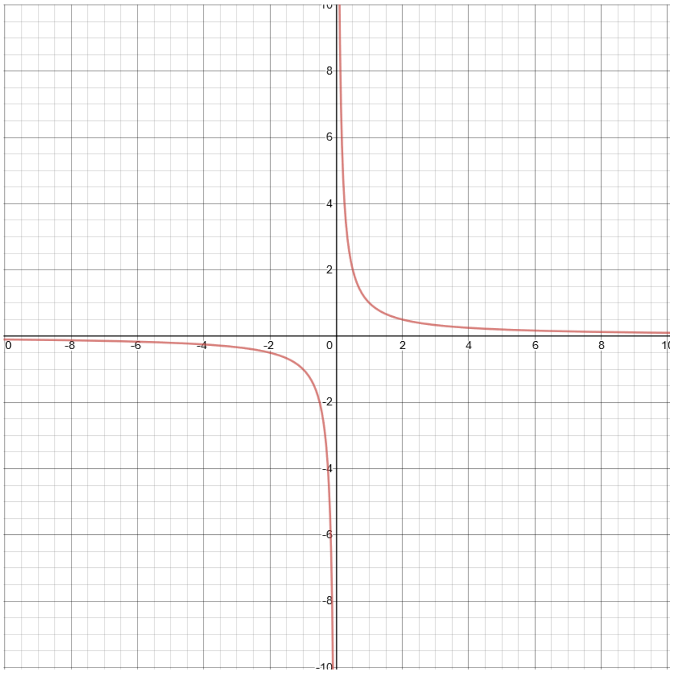
Re-writing in vertex form we get

$y = 2(x + 1.5)^2 + 35.5$

3

Rational Functions

Rational functions are functions which have $x$in the denominator. Moreover, $\frac{1}{x}$is a rational fraction.



Example 1 (Example 8, Page 88):

Determine the domain and range of the function $y = \frac{2x-1}{1-3x}$and write down the equations of the asymptotes and the coordinates of any axis intercepts.

We first equate the denominator to 0, getting the equation

$1 - 3x = 0$
Hence $x = \frac{1}{3}$is a vertical asymptote.

We then divide the highest power of $x$in the denominator to the numerator getting

$\frac{2x}{-3x} =- \frac{2}{3}$is a horizontal asymptote.

To find the $x$intercept we equate $y = 0$

$0 = \frac{2x-1}{1-3x}$

$0 = 2x - 1$

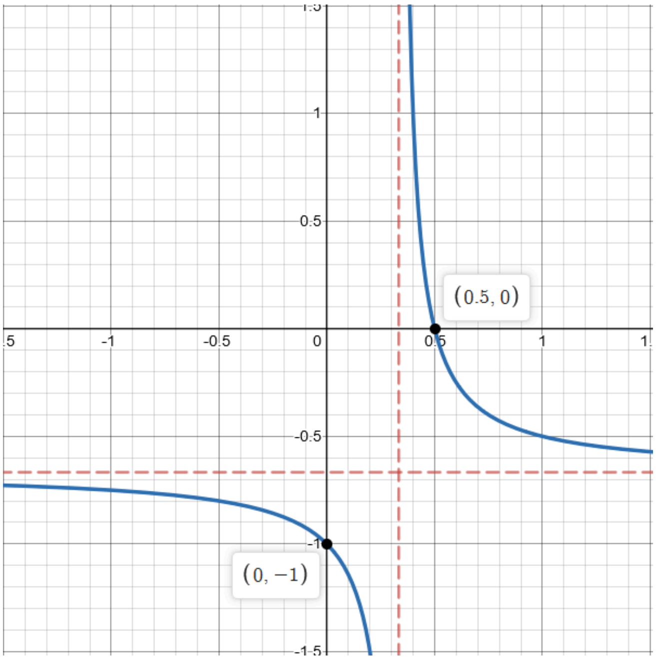
$x = 0.5$

To find the y intercept we equation $x = 0$

$y = \frac{-1}{1}$

$y =- 1.$

Sketch:



Hence we can write the domain as

$x \mid x \in R,\ x \neq \frac{1}{3}$

And the range as

$y \mid y \in R,\ y \neq - \frac{2}{3}$

4

Square root functions

Defined as $y = \sqrt{x}$
Not to confuse with $y^2 = x$ because $y = \pm\sqrt{x}$

Example 1 (Example 9, page 89):

Determine the domain and range of $y = 2 - \sqrt{2x + 3}$, the axis intercepts and describe the transformation from the graph $y = \sqrt{x}$ and confirm your answer graphically

We first know that the square root part of the function cannot be lower than 0. Hence we rewrite it as the following

$2x + 3 \geq 0$

$x \geq -\frac{3}{2}$

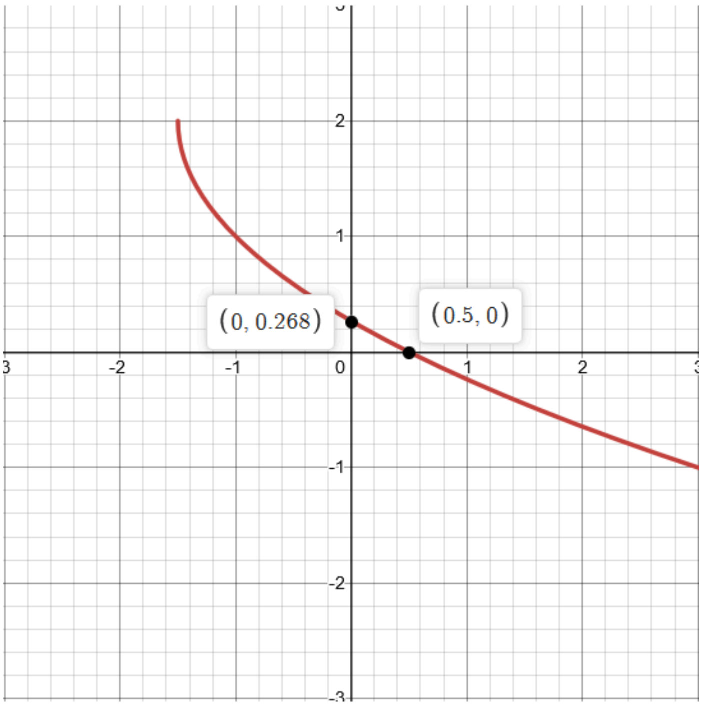
Since the smallest value that $\sqrt{2x + 3}$can achieve 0, the largest y value that can be achieved is 2.

$y \leq 2$

As such the domain and the range can be written as
$x \mid x \in R,\ x \geq -\frac{3}{2}$
$y \mid y \in R,\ y \leq 2$

The intercepts are the following:
When $y = 0$the $x$intercept is
$0 = 2 - \sqrt{2x + 3}$
$2 = \sqrt{2x + 3}$
$4 = 2x + 3$
$x = 0.5$

When $x = 0$ the intercept is
$y = 2 - \sqrt{3}$



Describing the transformation:

Since there is a negative sign there is a reflection on the $x$axis

Vertical translation of 2 up, otherwise as $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$

Horizontal stretch factor of 0.5

Horizontal translation of 1.5 left, otherwise as $\begin{pmatrix} -1.5 \\ 0 \end{pmatrix}$

5

Partial Fraction Decomposition

Partial fraction decomposition helps split up a whole quotient quadratic into different parts.

Example 1 (Example 12, Page 92):

Express $\frac{2x-5}{x^2-3x+2}$ in partial fractions

We first factorise the denominator

$\frac{2x-5}{(x-2)(x-1)}$

Then we write in the form

$\frac{2x-5}{(x-2)(x-1)} = \frac{A}{(x-2)} + \frac{B}{(x-1)}$

We multiple both sides by $(x - 2)(x - 1)$ and get

$2x - 5 = A(x - 1) + B(x - 2)$

Now to find values of $A$and Bwe substitute $x$for their solutions:

For example, substitute $x = 1$

$2 - 5 = A(0) + B(-1)$
$-3 = -B$
$B = 3$

Substitute $x = 2$

$4 - 5 = A(1) + B(0)$
$-1 = A$

Thus, we rewrite substituting $A$and $B$to get the final form:

$-\frac{1}{(x-2)} + \frac{3}{(x-1)} = \frac{2x-5}{x^2-3x+2}$

Partials of the form

$\frac{A}{(x+h)^k}$

Can be expressed as

$\frac{A}{(x+h)^k} = \frac{B}{(x+h)} + \frac{C}{(x+h)^2} + \cdots + \frac{D}{(x+h)^{k-1}} + \frac{E}{(x+h)^k}$

6

Quotient & Reciprocal Functions

Quotient functions are functions which are $\frac{1}{f(x)}$

General rules of quotient and reciprocal functions:

Zeros/solutions of $f(x)$ become the vertical asymptotes in $\frac{1}{f(x)}$

All important $y$values become $\frac{1}{y}$. This includes $y$intercepts,minimums and maximums of a function. They all become reciprocated.

When $f(x) > 0$, $\frac{1}{f(x)} > 0$. When $0 > f(x)$, $0 > \frac{1}{f(x)}$.

Example 1 (Example 31, Page 119):

Draw the graph of $y = x(x - 4)$. On the same set of axes, sketch the graph of the reciprocal. For both graphs, label any intercepts, zeros, extremas and asymptotes.

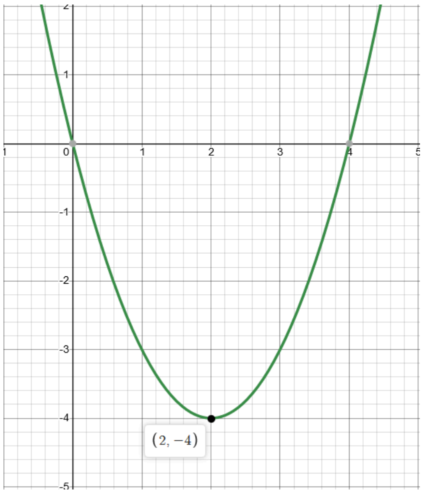
For the graph $y = x(x - 4)$, zeroes are at 0 and 4.

Knowing this, we know that the minima is at $x = 2$because of symmetry.

Hence, if we substitute to get the y coordinate it is at $(2, \ -4)$.

No asymptotes in a quadratic graph.

Hence we sketch:

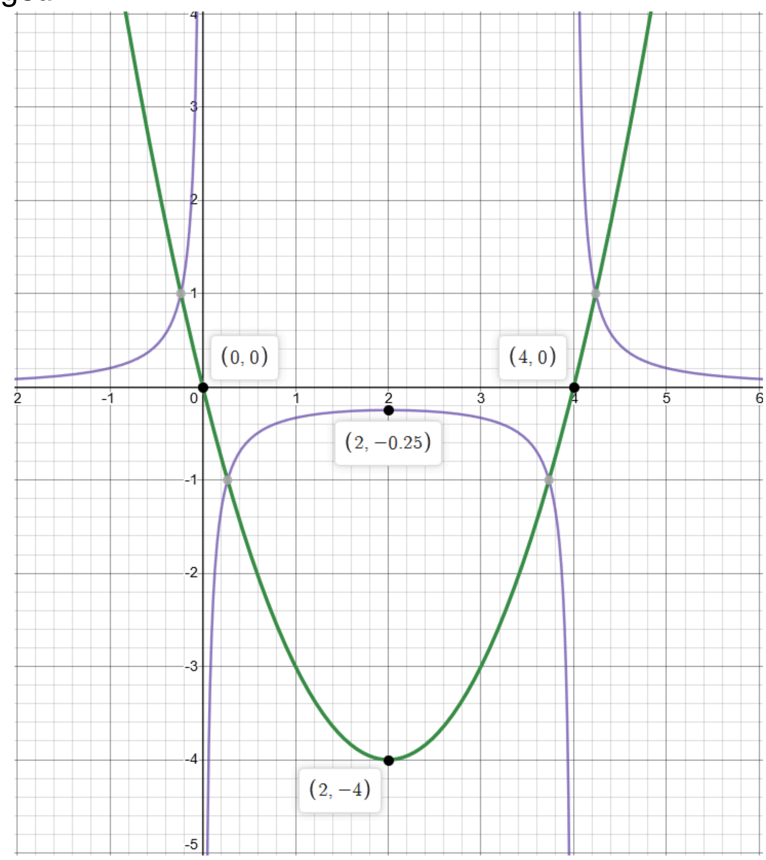


For the reciprocal graph, we know that vertical asymptotes exist at $x = 0,\ 4$ because $\frac{1}{f(x)}$, $f(x) \neq 0$.

We know the vertical asymptote exists at $y = 0$from the x^nasymptote power division rule.

The coordinates of the minima hence become $(2, -\frac{1}{4})$, which is now in fact a maxima.

Sketching we get:



Example 2 (Example 11, Page 91)

Find the domain and range of $y = \frac{1}{\sqrt{x+1}}$ and state any equation of asymptotes. Confirm your answers graphically.

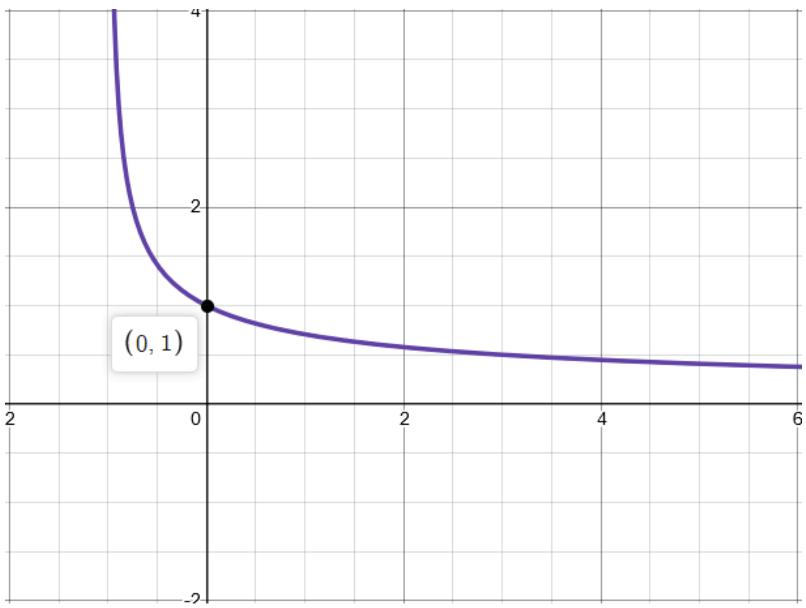
To find the vertical asymptote we equation the denominator to 0, hence

$\sqrt{x + 1} = 0$
$x = -1$

That means that x has to be $x > -1$ for it not to be imaginary numbers

Horizontal asymptote at $y = 0$ because the denominator can be extremely large, but the product of $y = \frac{1}{\sqrt{x+1}}$ will never reach 0 even if x is a very, very large number. Or, because there are no x powers present in the numerator, it is safe to say that $\frac{0}{x^n} = 0$

The range $y > 0$, $y \neq 0$ because both numerator and denominator will always result in a positive number.



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Absolute Value and Modulus Functions & Inequalities

Absolute value functions can never be negative.

To remember:

$y = a|x - h| + k$

Where $(h, k)$is the vertex of linear modulus graph

When solving modulus inequalities the best thing to do is to usually graph them.

When solving modulus inequalities in which the modulus function's gradient is a negative, swap the inequalities around when separating them. (Example 4)

For inequality questions, you can first solve it with an equal sign. After finding all the x values, you can apply logic to create the inequality in order not to confuse yourself when solving with inequalities.

Example 1 (Example 13, Page 95):

Determine the domain and range of $|2 - 2x| + 1$and sketch the function.

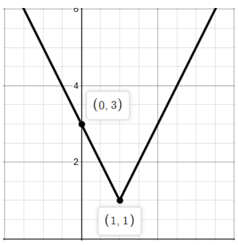
$|2 - 2x| = | - 2 \times (- 1 + x)| = |- 2| \times |- 1 + x|$
$= 2|x - 1| + 1$
Hence$(1, 1)$is the vertex
It is concave up because $a$is positive

$x$intercept when $y = 0$, at $2|x - 1| + 1 = 0$
$|x - 1| =- 0.5$ hence it doesn't exist

$y$intercept when $x = 0$, at $2|0 - 1| + 1 = 3$, $(0, 3)$

Domain$x \mid x \in R$
Range $y \mid y \in R,\ y \geq 1$

Sketch:



Example 2 (Example 16, Page 96):

Solve $|x + 1| =- 2x - 5$

We first need to rewrite the absolute value equation into 2 different equations
$x + 1 =- 1(- 2x - 5)$
$x + 1 =- 2x - 5$
Solving for the first
$x + 1 = 2x + 5$
$x =- 4$
Solving the second
$x =- 2$

We substitute back in both solutions to see if they work
$|- 4 + 1| = 8 - 5$
LHS = RHS, hence its a solution

$|- 2 + 1| \neq 4 - 5$
LHS ≠RHS, hence its not a solution

Example 3 (Example 17, Page 97):

Solve $|3x - 4| = |2x + 3|$
Let both sides be positive
$3x - 4 = 2x + 3$
$x = 7$
Let one side by negative and one positive
$- 3x + 4 = 2x + 3$
$x = \frac{1}{5}$

-OR-

ALTERNATIVE METHOD
Square both sides
$(3x - 4)^2 = (2x + 3)^2$
$9x^2 - 24x + 16 = 4x^2 + 12x + 9$
Make it a single quadratic
$5x^2 - 36x + 7 = 0$
Factorise
$(5x - 1)(x - 7)$

$x = \frac{1}{5}, 7$

-THEN-

We check through substitution.

$\left|\frac{3}{5} - 4\right| = \left|\frac{2}{5} + 3\right|$

$\frac{17}{5} = \frac{17}{5}$

LHS =RHS, hence its a solution

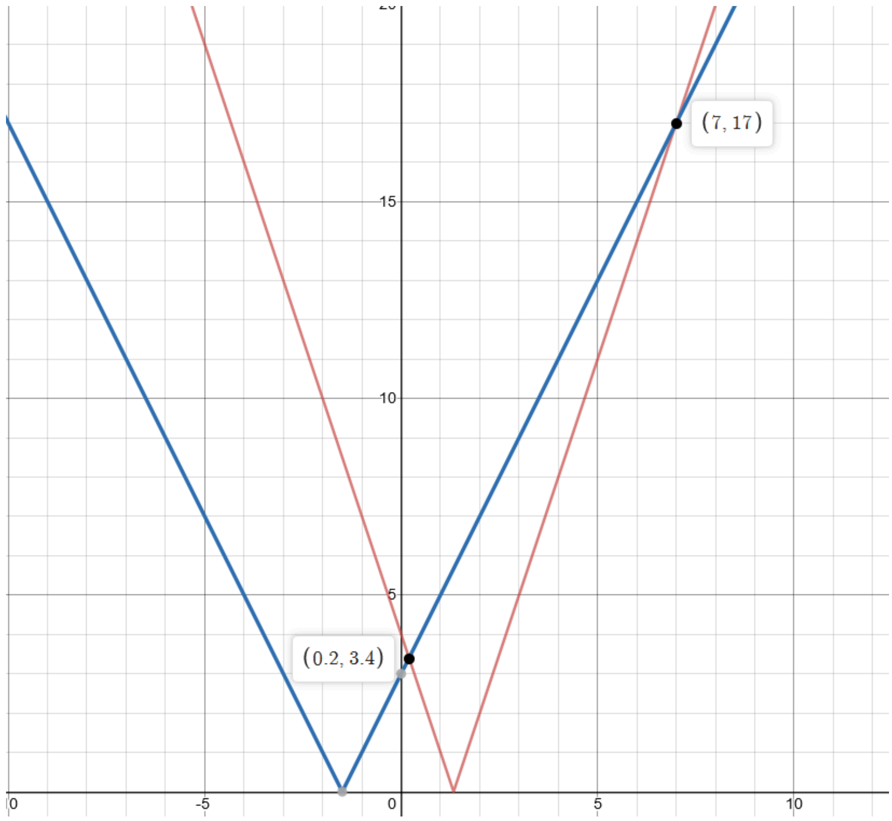
$|21 - 4| = |14 + 3|$

$7 = 7$

LHS = RHS, hence its a solution

OTHER METHOD

Graph it, find intersections



Example 4 (Example 18, Page 98):

Solve

a. $|2x - 4| < 4$

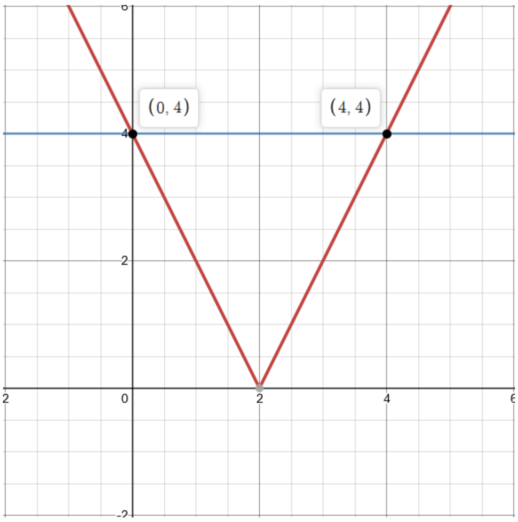
b. $|3 - 2x| \geq 1$

A.
Create $\pm$inequality
$-4 < 2x - 4 < 4$
Separate
$-4 < 2x - 4$
$0 < x$
Seperate the other
$2x - 4 < 4$
$x < 4$
Hence
$\therefore 0 < x < 4$

-OR-

Instead of doing inequality solving, solve it by finding $x$intersections (equating), and then apply logic for the inequality in the last step.

-OR-
ALTERNATIVE METHOD
Graph it, finding the intervals in which the line $y = 4$ is greater than the modulus function



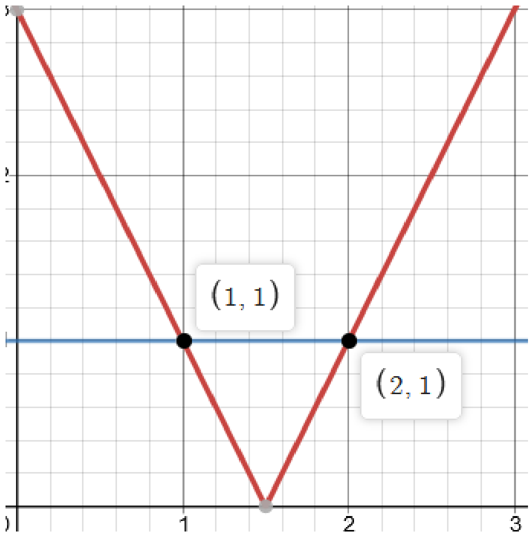
B.
Create $\pm$inequality (the inequalities in this example swap because the gradient of the graph is negative)
$3 - 2x \leq 1$
$3 - 2x \geq -1$
Separate
$3 - 2x \geq -1$
$-2x \geq -4$
$x \geq 2$

Seperate the other
$3 - 2x \leq 1$
$x \leq 1$
Hence
$\therefore x \geq 2, x \leq 1$

-OR-

Instead of doing inequality solving, solve it by finding $x$intersections (equating), and then apply logic for the inequality in the last step.

-OR-
ALTERNATIVE METHOD
Graph it, finding the intervals in which the modulus function is greater than $y = 1$ line



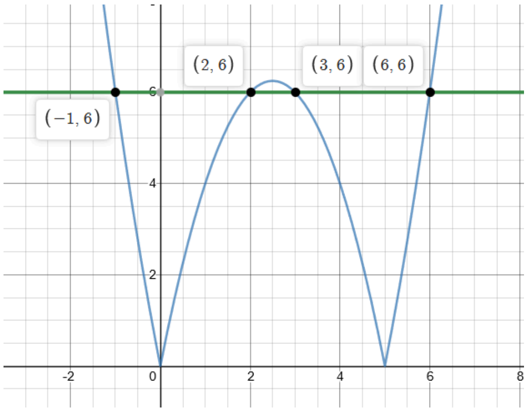
Example 5 (Example 19, Page 99):

Solve $\left|x^2 - 5x\right| < 6$

Create $\pm$inequality
$-6 < \left|x^2 - 5x\right| < 6$
Separate
$-6 < x^2 - 5x$
$0 < x^2 - 5x + 6$
$0 < (x - 3)(x - 2)$
Since this is the internal part of the quadratic and the inequality states that we're looking for the part under, we know that $x < 2, 3 > x$.
Separate other
$x^2 - 5x - 6 < 0$
$(x + 1)(x - 6) < 0$
Since this is the external part of the quadratic, and the inequality states that we're

looking for the part under, we know that $-1 < x < 6$.

Thus $-1 < x < 2,\ 3 < x < 6$.



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Piecewise defined functions

Piecewise functions are types of functions which have special properties in set domains. For example, for $x \leq 0 \; = \; 5x + 3, \; x > 0 = x^2$.

Example 1 (Exercise 2J, Page 102, 6b):

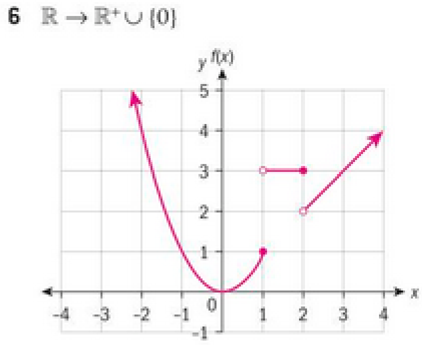
Rewrite $y = |3x - 9| + 2$ as a piecewise function

$|3||x - 3| + 2 = 3|x - 3| + 2$, hence inequality must be from $x = 3$, i.e $x \leq 3$ and $x > 3$as defined in the form $a|x - h| + k$ where $(h, k)$is a maxima or a minima

When defining ≤ of a modulus function as piecewise, set the modulus to negative, because it is a reflected line and thus is the opposite of the other.

Thus,
$f(x) = \{x \leq 3, \; - 3x + 11$
$\{\, x > 3, \; 3x - 7$

Example 2 (Exercise 2K, Page 106, 6):
Determine whether the function is onto, one-to-one, neither or both.



This function is an onto function, since R, has R^+U. This is, however, not a one to one function because for say $y = 1$, then $x = - 1, \; 1$, hence many to one.

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Odd and Even functions

A function can be odd, even or neither. This can be determined by doing the following:

If it is even, then $f(x) = f(-x)$.
If it is odd, then $-f(x) = f(-x)$.

An even function is symmetrical about the y axis.
An odd function has rotational symmetry of 180° from the origin.

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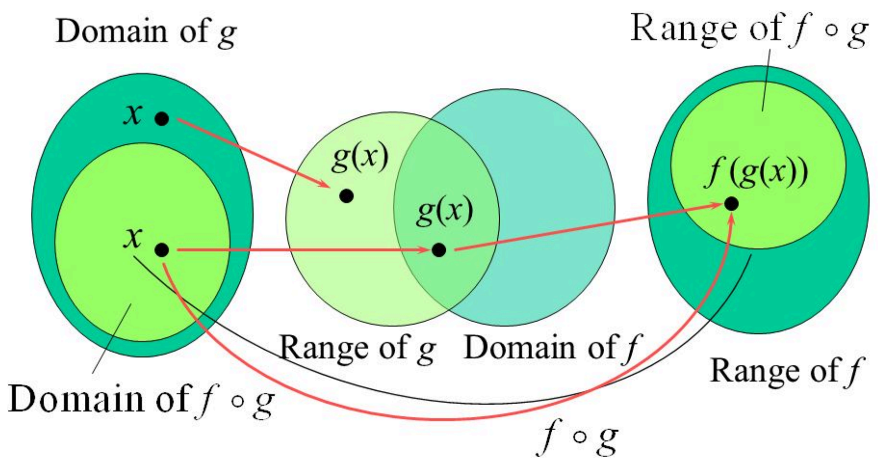
Function composition

Function composition is when 2 functions are combined. This is otherwise stated as, for example, $f(g(x))$, otherwise as $(f \circ g)(x)$.



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Hence or otherwise, the newly defined function's domain can be found using the properties stated above, or, can be done so through the use of logic of the new function.

Example (from IB discord, source unknown):

Suppose $f(x) = 9 - \sqrt{x}$ and $g(x) = x^2 + 4$.

a. Find $(f \circ g)(x)$'s asymptotes
b. Hence or otherwise, state its domain and range.

A.
The function has slanted asymptotes.

B.
$(f \circ g)(x) = 9 - \sqrt{x^2 + 4}$.

We first notice that inside the square root we cannot have negative numbers. However, since it is x^2, we also notice that all negatives become positives. Hence, the inside can never become a negative. Therefore $x \in R$ is the domain.

For range, let the function be defined as $9 - a$ where $a = \sqrt{x^2 + 4}$. From this we could deduce that the maximum it can take is 9, and it is decreasing from nine. However, the minimum that a can take is when $x = 0$, i.e $\sqrt{4} = 2$. As such, it will always be minused by 2 or more. Hence we can rewrite as $9 - (2 + b)$, where $b \in R$. Thus the range is $y \leq 7,\ y \in R$.

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Inverse Functions

Inverse functions are noted as $f^{-1}(x)$. The functions that are inverted go through a transformation of a reflection upon the line $y = x$. This can, as a result, make some inverse functions actually not a function because they may not pass the vertical line test. I.e $f(x) = x^2$.
In inverse functions the domain and range swap around.

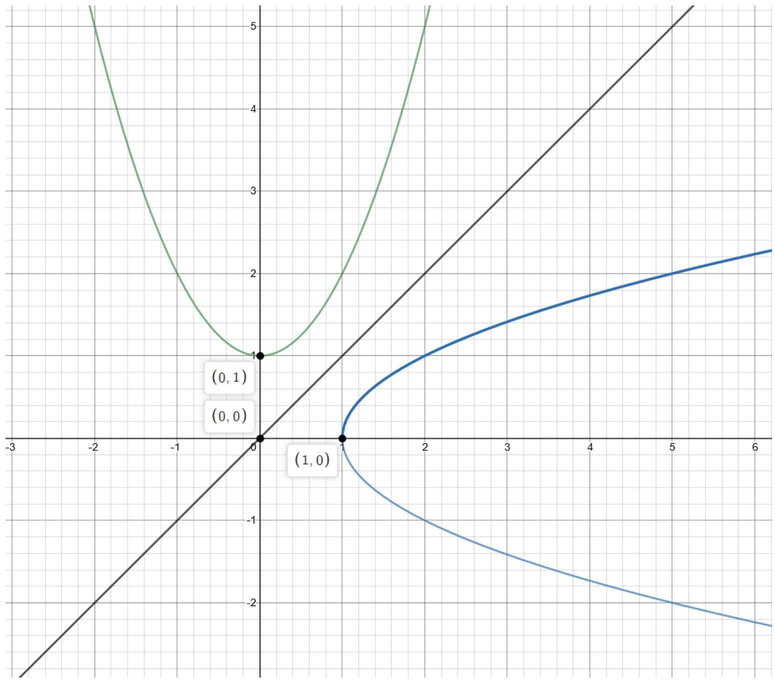
Example (Example 29, Page 115):

a. Find the inverse relation of $y = x^2 - 1$, and graph both the function and its inverse relation on the same set of axes

b. State two different domain restrictions of the function, and the corresponding ranges, in order that its inverse is a function, and for each, state the domain and range of the inverse function

c. State the two functions, with their restricted domains, and their corresponding inverse functions.

A.
$x = y^2 + 1$
$y = \pm\sqrt{x - 1}$



B. We need to limit the original function so that the inverse is also a function. Thus, we can limit that $0 \leq x$ or $x \leq 0$. The range can stay as it is, and as such it is $1 \leq y$.

The new function's range would be defined as $y \leq 0$if the original function is defined as $x \leq 0$. If $0 \leq x$ then $0 \leq y$. The domain is $1 \leq x$.

C. If $x \leq 0$, then the inverse would be $y =- \sqrt{x - 1}$
If $0 \leq x$, then the inverse would be $y = \sqrt{x - 1}$

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Translations

Translations are what would change the original graph into something new.

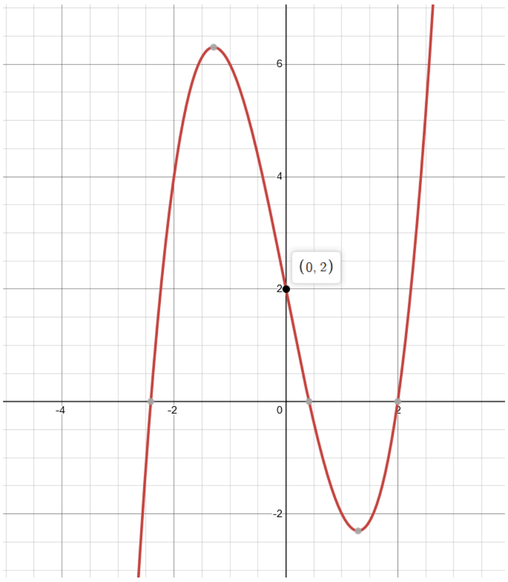
General tips:

Order of transformations:

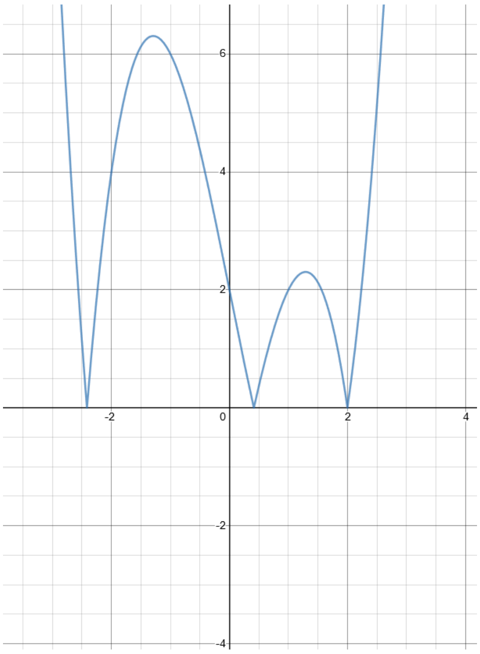
http://mathforum.org/library/drmath/view/68503.html

Absolute value translations

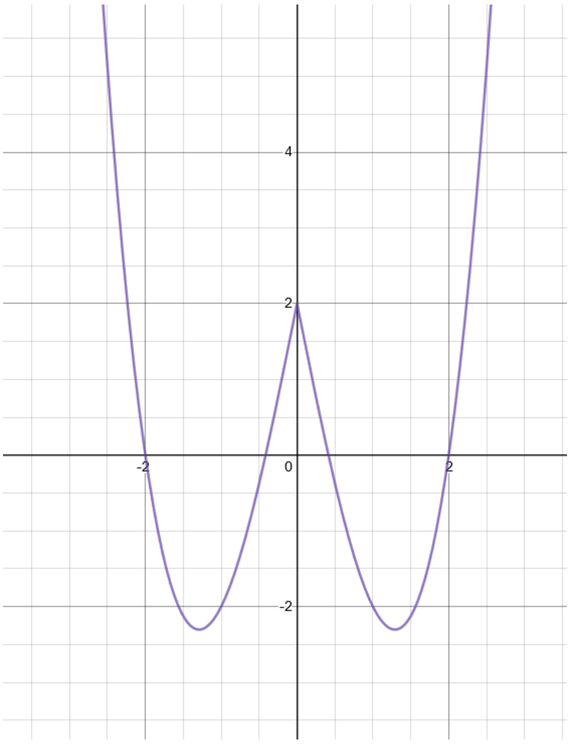
Let a function be defined as $f(x) = x^3 - 5x + 2$.



If $|f(x)|$, then all the -y values will become positive, as such it bounces back off.



If $f(|x|)$, then all the input negative x become as output as if they were positive x values. As a result, it creates symmetry. We still input $-x$ values, as such the domain for $x \le 0$ still exists, it's just that it will resemble the positive x values in the result.



Reciprocals

To graph the reciprocal of $y = f(x)$:

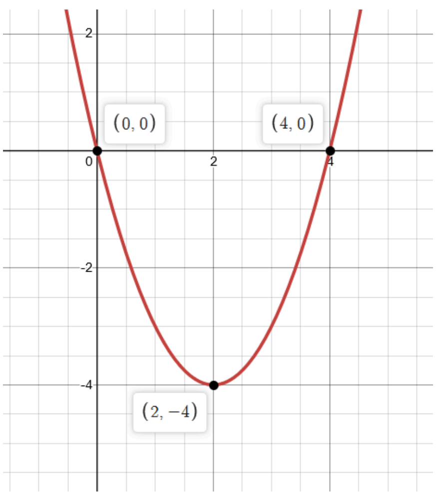
- Where they exist, the zeros of $y = f(x)$ are the vertical asymptotes of $y = \frac{1}{f(x)}$.
- If $y = b$ is the y-intercept of $y = f(x)$, then $y = \frac{1}{b}$ is the y-intercept of $y = \frac{1}{f(x)}$.
- The minimum value of $y = f(x)$ occurs at the same value of x as the maximum of $y = \frac{1}{f(x)}$ and vice versa.
- When $f(x) > 0$, $\frac{1}{f(x)} > 0$; when $f(x) < 0$, $\frac{1}{f(x)} < 0$.
- When $y = f(x)$ approaches 0, $\frac{1}{f(x)}$ will approach $\pm\infty$, and vice versa.

Example 1 (Example 31, Page 119):

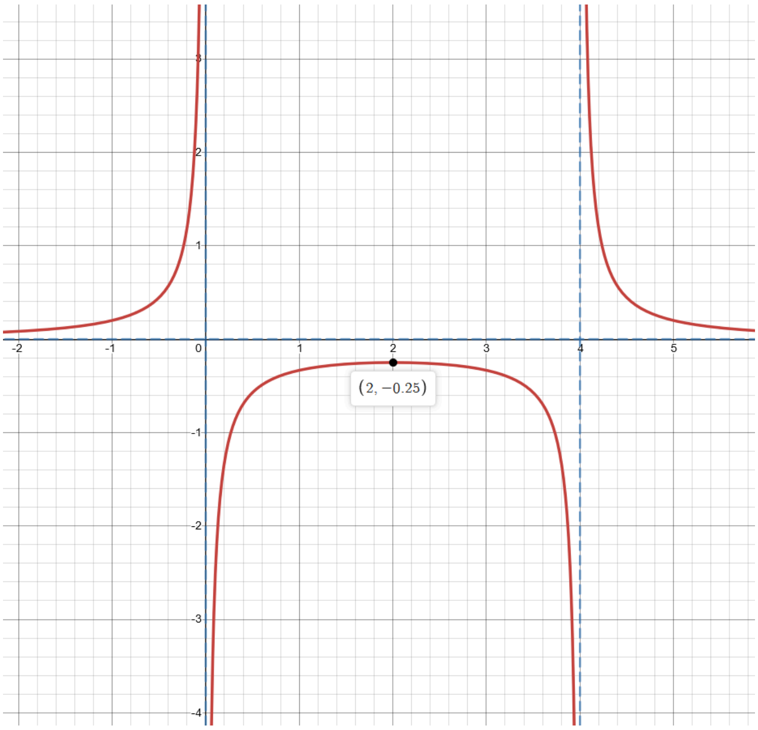
Draw the graph of $y = x(x - 4)$. On the same set of axes, sketch the graph of its reciprocal. $y = \frac{1}{x(x-4)}$. For both graphs, label any intercepts, zeros, extrema and asymptotes.

x zeros are at $0, 4$.
Using formula $\frac{-b}{2a}$ (SL 2.6) which determines the x coordinate of the maxima/minima we get $x = 2$. Substituting $x = 2$ to get y we get $y =- 4$.

y intercept is at 0 because when $x = 0,\ y = 0$.

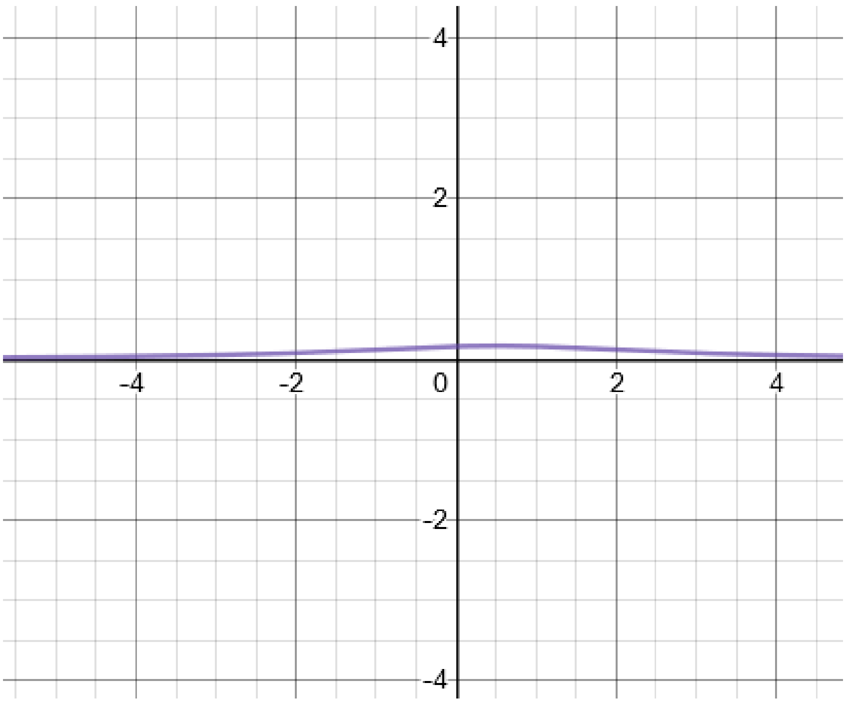


For $y = \frac{1}{x(x-4)}$ we know that the denominator can't be zero, hence at $x = 0,\ 4$ there are vertical asymptotes. Furthermore, for the y value of the minima we reciprocate to get $-\frac{1}{4}$ as the new maxima. The Horizontal asymptote by $\lim_{h \to 0} = 0$, as the top has x with coefficient 0. The general shape is as follows:



Example 2 (made-up):

Suppose that the quadratic then has imaginary roots, as such it has no asymptotes. Then, the general shape of such would follow:



Where the peak of the function is the $\frac{1}{y}$value of the original function. The Horizontal asymptote by $\lim_{h \to 0} = 0$, as the top has x with coefficient 0.

Stretch factors

Stretch factors multiply all the values out. In x values the stretch factor is the opposite.

I.e $f(x) = x^2 + x + 1$, then $2(x^2 + x + 1)$, there is horizontal a stretch factor of 2 (all y values are $\times\ 2$)

If $f(2x)$, then there is a vertical stretch factor of $\frac{1}{2}$ (all x values are $\times\ \frac{1}{2}$)

Horizontal & vertical translations are translations which move the graph up right left down.

I.e $f(x) = x^2 + x + 1$

$f(x) + 1$, then there is a vertical translation of 1 unit up.

$f(x + 1)$, then there is a horizontal translation of 1 unit left.

Reflection transformations

I.e $f(x) = x^2 + x + 1$
$f(- x)$ is a reflection on the y axis.
$- f(x)$ is a reflection on the x axis.

Combination and orders

Example 5: Graph by hand: $f(x) = 0.5(3x^3 + 9) - 1$

Parent function: $y = x^3$

Using the formula $a\,f(b(x + c)) + d$ for separations, we get $f(x) = 0.5(3(x^3 + 3)) - 1$

But again, the inner most parentheses in this problem does not indicate a horizontal shift since the power of x is not 1. This inner most parentheses is indicating a vertical shift.

Follow the order of operations to prepare the graph.

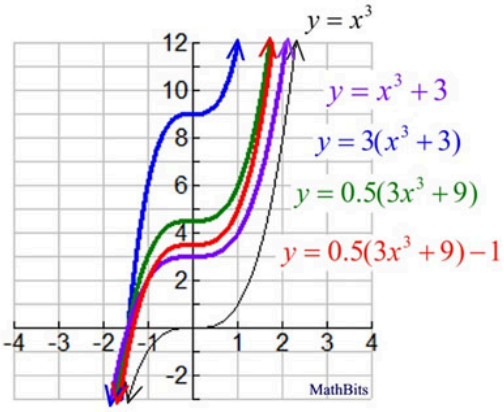
Step 1: (in purple)
Do the inner most parentheses first,
$y = x^3 + 3$ which shifts the graph 3 units up.

Step 2: (in blue)
Apply a vertical stretch of 3.
$y = 3(x^3 + 3)$ which multiplies y-values times 3.

Step 3: (in green)
Apply a vertical stretch of 0.5.
$y = 0.5(3(x^3 + 3))$ which multiplies y-values times ½.

Step 4: (in red)
Apply the vertical shift of -1,
$y = 0.5(3(x^3 + 3))$ -1 which shifts graph 1 unit down.

Final graph in red.



I.e in $f(ax + b)$ deal with the horizontal translation first, and then apply scale factor

Proof:
Since $f(ax + b)$ is a translation which is inside the brackets, then we can define it as different functions. Let $k(x) = f(ax + b)$. When we split the function up, to get $g(x) = f(ax)$ and $h(x) = f(x + b)$. If we do the stretch first, then $g(h(x))$ to get $f(a(x + b)) = f(ax + ab) \neq k(x)$. If we do horizontal translation first, then $h(g(x)) = f(ax + b) = k(x)$.

In $af(x) + b$ apply the scale factor, and then the vertical shift.

Proof:
Here, nothing is put into brackets thus it follows regular rules. It can otherwise be re-written as $ay + b$ where $f(x) = y$
Translations.pdf

Topic 3 Complex Numbers

1

Completing the square

Completing the square helps you put a quadratic into vertex form $(x - h)^2 + k$ where (h, k) are the coordinates of the minima/maxima.

Example 1 (Exercise 3B, d):

Complete the square and give the answer in exact form

$3x^2 - 7x + 2 = 0$

If there is a coefficient to the x^2, divide it from x^2 and x

$3(x^2 - \frac{7}{3}x) + 2 = 0$

Divide the coefficient of x by 2. Remove the x from each side and put the square outside the bracket

$3(x - \frac{7}{6})^2 + 2 = 0$

Square the term without the x and put it outside the squared bracket with a minus

$3(x - \frac{7}{6})^2 - \frac{7^2}{6^2}) + 2$

$3(x - \frac{7}{6})^2 - \frac{49}{36}) + 2$

Times it by the coefficient we divided in the initial step

$3(x - \frac{7}{6})^2 + 2 - \frac{147}{36}$

$3(x - \frac{7}{6})^2 - \frac{25}{12}$

To solve, now we re-arrange it so that

$3(x - \frac{7}{6})^2 = \frac{25}{12}$

$(x - \frac{7}{6})^2 = \frac{25}{36}$

$(x - \frac{7}{6}) = \pm \sqrt{\frac{25}{36}}$

$(x - \frac{7}{6}) = \pm \frac{5}{6}$

Now to solve for the plus and the minus

$x - \frac{7}{6} = - \frac{5}{6}$

$x = \frac{1}{3}$

And

$x - \frac{7}{6} = \frac{5}{6}$

$x = 2$

2

Quadratic Formula

The quadratic formula is used to solve a quadratic.

$x = \frac{b \pm \sqrt{b^2 - 4ac}}{2a}$(SL 2.7)

Example (Example 4, C, Page 154):

Solve using the quadratic formula:

$5x^2 - 2x - 1 = 0$

First find the terms $ax^2 + bx + c$, so $a = 5,\ b =-\ 2,\ c =-\ 1$.Plug and solve

$$x = \frac{-2 \pm \sqrt{(-2)^2 - 4 \times 5 \times -1}}{2 \times 5}$$

$$x = \frac{-2 \pm \sqrt{4 + 20}}{10}$$

$$x = \frac{-2 \pm \sqrt{24}}{10}$$

$$x = \frac{-2 \pm 2\sqrt{6}}{10}$$

$x = \frac{-1 + \sqrt{6}}{5}$ or $x = \frac{-1 - \sqrt{6}}{5}$

3

Discriminant Formula

The discriminant formula helps determine whether a quadratic has solutions.

Let us denote that $\Delta$is the discriminant formula.

If $\Delta > 0$, then the quadratic has 2 distinct real roots.

If $\Delta = 0$, then the quadratic has 1 real repeated root

If $\Delta < 0$, then the quadratic has no real roots, but 2 complex ones.

$\Delta = b^2 - 4ac$ (SL 2.7)

Example (Example 8, Page 158):

Find the values of r for which the equation $x^2 + 3rx + 1 = 0$ has

a) Two real solutions
b) On real solution
c) No real solutions

A:

$b^2 - 4ac > 0$

$9r^2 - 4 \times 1 \times 1 > 0$

$r^2 > \frac{4}{9}$

$r < -\frac{2}{3}, r > \frac{2}{3}$

B:

$9r^2 - 4 = 0$

$r^2 = \frac{4}{9}$

$r = \pm\frac{2}{3}$

C:

$9r^2 - 4 < 0$

$r^2 < \frac{4}{9}$

$|r| < \frac{2}{3}$

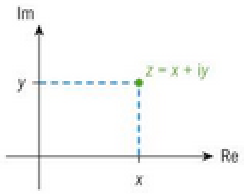
$-\frac{2}{3} < r < \frac{2}{3}$

4

Complex numbers

Complex pair z is denoted as $z = a + bi$, where $a, b \in R$and $i \in C$. The real part is seen as the a of the equation, whilst the imaginary part is seen as bi.

The modulus of a complex pair z is
$|z| = \sqrt{a^2 + b^2}$(the distance between the origin and the pair z in the real and imaginary plane)



The complex conjugate form of a complex number is denoted as z^*, let $z = a + bi$ then $z^* = a - bi$. The sign for the complex part swaps.

5

Rational Roots Theorem

6

7

8

Topic 4 Differentiation

1

Introduction to limits

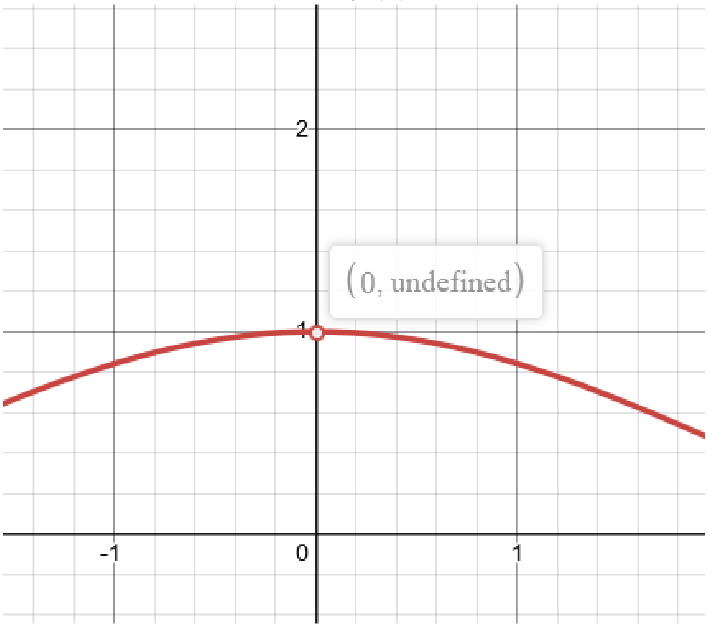
Differentiation is the calculus of limits. Limit is to be thought of the result as approaching specific value, let's call it a. This can be denoted as the following

$$\lim_{x \to a} f(x)$$

The following rules are essential for limit to exist:
The limit $\lim_{x \to a} f(x) = L$ exists if and only if $\lim_{x \to a^-} = L$ and $\lim_{x \to a^+} = L$
Discontinuity at point L is not a concern, consider the famous example

$$f(x) = \frac{\sin x}{x}$$

There is discontinuity at $x = 0$.
However, consider the following graph of $f(x)$:



Even then, it is evident that limit exists as $\lim_{x \to 0} \frac{\sin x}{x} = 1$, because $\lim_{x \to 0^+} \frac{\sin x}{x} = 1$ and $\lim_{x \to 0^+} \frac{\sin x}{x} = 1$.

2

Limits of infinity and asymptotes

Limits can also be used to find asymptotes of graphs. Consider the following polynomial function $f(x)$ defined as

$f(x) = \frac{x^2+x+1}{x^2-x-1}$

Then, we know that in horizontal asymptotes, $x \to \infty$.
Hence, consider

$\lim_{x\to\infty} \frac{x^2+x+1}{x^2-x-1}$

In order to evaluate this, divide everything by x^2

$\lim_{x\to\infty} \frac{1+\frac{1}{x}+\frac{1}{x^2}}{1-\frac{1}{x}-\frac{1}{x^2}}$

And evaluating this limit we obtain 1. Which means that at $y = 1$ our x approaches infinity, that is a horizontal asymptote.

In order to find our vertical asymptotes, we can consider parts where our denominator is not defined. That is, $\frac{1\pm\sqrt{5}}{2}$. Hence these are the vertical asymptotes.

Moreover, it is important to know the following

$\lim_{n\to\infty} (1 + \frac{1}{n})^n = e$

More specifically,

$\lim_{n\to\infty} (1 + \frac{x}{n})^{yn} = e^{xy}$

3

Continuity

Continuity of a function is when you actually can draw the function without having to lift your pen, to simply put. In mathematical terms:
A function is defined to be ‘continuous’ if and only if the following things hold true $\lim_{x \to a} = L$, which implies $\lim_{x \to a^+} f(x) = L$ and $\lim_{x \to a^-} f(x) = L$, where $f(a) = L$.

4

Differentiation

Differentiation is finding the rate of change at an infinitesimal point. Consider the definition of first principles, which is given by:

$$\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

Derivation:
The average rate of change can be given by

$$\frac{f(x)-f(a)}{x-a}$$

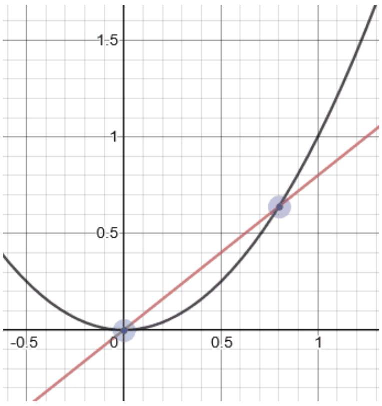
From the slope formula $\frac{y_2-y_1}{x_2-x_1}$
Let $h = x - a$, then $x = h + a$
Hence we obtain

$$\frac{f(a+h)-f(a)}{h}$$

However, since we are looking for the instantaneous rate of change, that is infinitesimal rate of change, we then want to minimise the gap between $x - a$. So we evaluate the limit

$$\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$$

Consider the following diagram:



It is then possible to see in the diagram that as the gap approaches 0 between the 2 intervals the gradient becomes what it is at that point.

And hence arises some differentiation rules, such as the power rule

Consider $f(x) = x^n$ from some $x \in Z$

Then, by first principles

$$\lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$$

We binomially expand to obtain

$$\lim_{h \to 0} \frac{x^n + \binom{n}{1}x^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \cdots - x^n}{h}$$

Simplifying

$$\lim_{h \to 0} nx^{n-1} + \frac{n(n-1)}{2!}x^{n-2}h + \cdots$$

Finally we obtain

$$= nx^{n-1}$$

Although I won't go through the proofs of the following rules, they can be commonly found in the internet

Chain rule:

$\frac{dy}{dx} = \frac{dy}{df} \times \frac{df}{dx}$ or $d(f(g(x))) = f'(g(x)) \times g'(x)$

Product rule:

$d(f(x)g(x)) = f'(x)g(x) \times g'(x)f(x)$

Quotient rule:

$d(\frac{f(x)}{g(x)}) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$

5

Differentiability

Whilst it may be easy to guess that if the limit exists at all points of a function, then the function is differentiable. This is dire wrong. While it may show differentiability, it is not a sufficient condition. Consider the famous example:

$$f(x) = |x|$$

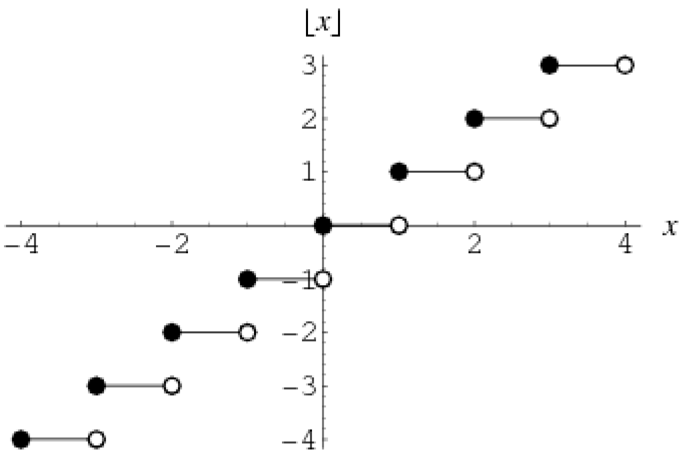
At $\lim_{x \to 0} f(x) = 0$, given that $\lim_{x \to 0^+} f(x) = 0$ and $\lim_{x \to 0^-} f(x) = 0$. However, consider first principles

$\lim_{h \to 0} \frac{|x+h|-|x|}{h}$

Which leads to
$\lim_{h \to 0} \frac{|x|+|h|-|x|}{h}$

Which implies
$\lim_{h \to 0} \frac{|h|}{h}$, notice that $\lim_{h \to 0^+} \frac{|h|}{h} = 1$ whilst $\lim_{h \to 0^-} \frac{|h|}{h} =- 1$, hence limit does not exist.
So at $x = 0$ it is not differentiable.

Another famous example would be the floor function denoted as
$f(x) = \lfloor x \rfloor$
Which looks like the following:



It is not differentiable at any $x \in Z$.

6

Tangents and Normals

A tangent of a function is the rate of change of the function at that point, that is slope m. A normal would be $-\frac{1}{m}$, that is a line that intersects 90° a point x.

Consider the following example:

Example (Page 248, Exercise 4K, 4):

Find the equation of normal to the curve $f(x) = \frac{8}{4+x^2}$ at $x = 1$. This curve is known as the "Witch of Agnesi"

Let us rewrite $f(x)$ as
$f(x) = 8(4 + x^2)^{-1}$
Differentiating using the chain rule we obtain
$\frac{dy}{dx} =- 16x(4 + x^2)^{-2}$. At $x = 1$ we obtain that $\frac{dy}{dx} =- \frac{16}{25}$. However, we are looking for the normal, thus $m = \frac{25}{16}$. At $f(1) = \frac{8}{5}$. And hence, using point slope form we can obtain that
$y - \frac{8}{5} = \frac{25}{16}(x - 1)$

7

Turning points

Turning points are maximums, minimums and points of inflection of a function.

Lemma 1:
Maximas and minimas of a function are given by
$f'(x) = 0$

Lemma 2:
$f''(x) = 0$ at $x = L$ is a necessary but not a sufficient condition to show that at point L a point of inflection exists.

For a point of inflection to exist, it must follow that
$f''(L + \varepsilon) =\pm$
$f''(L - \varepsilon) =\mp$, where ε is a positive small perturbation

If
$f''(L + \varepsilon) =-$
$f''(L - \varepsilon) =-$
Or
$f''(L + \varepsilon) =+$
$f''(L - \varepsilon) =+$
Then it is called a point of undulation.

(You may want to consider the graph of $x^4 - ax$ where a is real number and see that it there is a point of undulation at $x = 0$).

Lemma 3:
For any $f'(x) = 0$ at $x = L$, if
$f''(L) < 0$, then it is a maximum
$f''(L) > 0$, then it is a minimum

Lemma 4:
Both Lemma 4 and Lemma 2 are sufficient conditions to find the nature of turning points.

If $f'(L) = 0$
And if for $f'(L + \varepsilon) =+$
And $f'(L - \varepsilon) =-$
For some positive small perturbation ε
Then it follows that at $x = L$ is a minimum

If for $f'(L + \varepsilon) =-$
And $f'(L - \varepsilon) =+$
For some positive small perturbation ε
Then it follows that at $x = L$ is a maximum

8

Implicit differentiation

Implicit differentiation is seeing the change of y in terms of x of a function or a functional.

There are 2 IB ways of implicitly differentiating and finding the $\frac{dy}{dx}$ and one non-IB but faster method, if you know the multivariable chain rule of a functional. This will be covered in bonus material of this ‘flashcard’.

1st IB way (for the proof of this you may want to consider BONUS material just under this flashcard, as the non-IB way are interlinked with this method):

For example, let us consider the equation of a unit circle given by $x^2 + y^2 = 1$, then by differentiating

For x^2 we obtain $2x$
For y^2 we obtain $2y\frac{dy}{dx}$
For 1 we obtain 0

And thus our new equation is
$2x + 2y\frac{dy}{dx} = 0$, and solving for $\frac{dy}{dx}$
$\frac{dy}{dx} = \frac{2x}{-2y}$

2nd IB way
For example, let us consider the equation of a unit circle given by
$x^2 + y^2 = 1$, then by differentiating

For x^2 we obtain $2xdx$
For y^2 we obtain $2ydy$
For 1 we obtain 0

And thus our new equation is
$2xdx + 2ydy = 0$
Solving for $\frac{dy}{dx}$
$2xdx =- 2ydy$
$\frac{2x}{-2y} = \frac{dy}{dx}$

They are the same thing, however, Method 1 can be significantly easier at times. Nonetheless, 2nd IB way can also be used to do a lot of proofs which can be found at the other BONUS section (that is, the end of topic 4).

BONUS

Proof of IB first way and the Non-IB way
Consider the functional
$f(x, y) = x^2 + y^2 = 1$ where x and y are functions of t, that is $x(t)$ and $y(t)$.
Then, if we want to find $\frac{df}{dt}(x, y)$, we can consider for each individual case using the chain rule

For x^2we obtain $2x\frac{dx}{dt}$
For y^2 we obtain $2y\frac{dy}{dt}$
For 1 we obtain 0

And thus, we obtain that
$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

However, notice that if we do $\frac{\partial f}{\partial x} = 2x$ and $\frac{\partial f}{\partial y} = 2y$, and we can substitute these in to obtain

$$\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = 0$$

What if, we looked for change in x rather than change in t? Then we can just swap dt with dx. So we get (if you were to replace the partials with their values respectively you would achieve the same result as IB method 1. So it is actually multivariable chain rule in disguise, the total derivative of a functional with respect to x).
$$\frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0$$

Which simplifies to

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0$$

Solving for $\frac{dy}{dx}$
$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

Which is evident, test it for yourself for any multivariable equation.

9

Optimisation

10

Related Rates

11

Sketching Derivatives

12

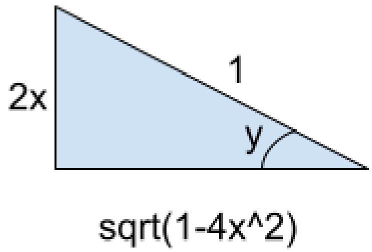
BONUS

NOTE: ALWAYS USE THE LOGARITHMIC $\ln$ operator to remove exponentials when differentiating, and then implicitly differentiate! Examples can be found for the proof of a^x below and the power tower in the EXTRAS section.

Proof of some standard HL derivatives

Find $\frac{d}{dx}\arcsin\ 2x$

Let $y = \arcsin\ 2x$
Then $\sin y = 2x$
Implicitly differentiating to obtain
$d(\sin y = 2x)$
$dy \times \cos y = 2dx$
Rearranging to obtain
$\frac{dy}{dx} = \frac{2}{\cos y}$
We can construct a triangle knowing $\sin y = 2x$, where SOH, O= $2x$ and H= 1



Hence we can obtain $\cos y$

$\cos y = \frac{\sqrt{1-4x^2}}{1}$

Substituting back into our equation

$\frac{dy}{dx} = \frac{2}{\sqrt{1-4x^2}}$

This method can always be used to prove standard derivatives.

Find $\frac{d}{dx}a^x$

Let $y = a^x$

Using ln operator

$\ln y = x \ln a$

Implicitly differentiating

$\frac{dy}{y} = dx \ln a$

Rearranging

$\frac{dy}{dx} = y \ln a$

Substituting y back in

$\frac{dy}{dx} = a^x \ln a$

Find $\frac{d}{dx}\log_a x$

Using change of base to get $\frac{\ln x}{\ln a}$

Differentiating to obtain

$\frac{d}{dx}\frac{\ln x}{\ln a}$

$= \frac{1}{x \ln a}$

Topic 5 Statistics and Probability

1

2

The quadratic formula is used to solve a quadratic.

3

4

5

6

7

Topic 6 Trigonometry

1

2

The quadratic formula is used to solve a quadratic.

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7

Topic 7 Exponents, Logarithms and Integration

1

Introduction to integrals

Integrals, or anti-derivatives, allows you to do the opposite of differentiation. It can be used to find, for example, the area of something under a graph by defining limits.

Generalisations include but are not limited to the following:

$\int f(x) \pm g(x) = \int f(x) \pm \int g(x)$ (Not in the formula booklet)

$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$(SL 5.5)

$\int (f(x) + b)^n \, dx = \frac{(f(x)+b)^{n+1}}{f'(x)\times(n+1)} + C$where $f(x) = ax^n$and where a, n are constants (Not in the formula booklet)

$\int \frac{1}{x} dx = \ln|x| + C$(SL 5.10)

$\int \sin x \, dx = -\cos x + C$(SL 5.10)

$\int \cos x \, dx = \sin x + C$(SL 5.10)

$\int e^x dx = e^x + C$(SL 5.10), however $\int e^{f(x)} dx = \frac{e^{f(x)}}{f'(x)} + C$. (Not in the formula booklet).

$\int a^x dx = \frac{1}{\ln a} a^x + C$(HL 5.15)

$\int \frac{1}{a^2+x^2} dx = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$(HL 5.15)

$\int \frac{1}{\sqrt{a^2-x^2}} = \arcsin\left(\frac{x}{a}\right) + C, \ |x| < a$(HL 5.15)

$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$or $\int u\, dv = uv - \int v\, du$(HL 5.16)

$\int \sec^2 x\, dx = \tan x$(Not stated as an integral, found in (HL 5.15))

$\int \sec x \tan x\, dx = \sec x$(Not stated as an integral, found in (HL 5.15))

$\int - \csc x \cot x = \csc x$(Not stated as an integral, found in (HL 5.15))

$\int \frac{f'(x)}{f(x)} = \ln(f(x)) + C$(Not in the formula booklet)

It is also important to use trigonometric identities when solving integrals. It is especially important to note that $\tan$ should be changed to $\sec$ whenever possible because of its nature.

$\log$integral questions should go through a base change to e, in order to make the question solvable by $\ln$ rules.

$n^{f(x)}$questions should make $u = f(x)$ and do integration by substitution. After getting the integral n^u, apply AHL 5.15 for a^x.

It is very important that if stumbled upon an $\int \cos^2 x$ or $\int \sin^2 x$, then use the double angle identity to change it to $\cos 2x$ otherwise integration is not possible!! (It is but not with IB knowledge, reduction formula)

If dealing with an integral with a large polynomial on the denominator, look into factorising and then apply partial fractions. If it cannot be factored, put it into vertex form and then apply a trigonometric substitution.

Example (Exercise 7A, Page 451, 12):

Given that $f'(\theta) = 4\sin(\theta + \frac{\pi}{4})\cos(\theta + \frac{\pi}{4})$ find the function $f(\theta)$ given that $f(0) = 1$.

We first use the identity that $\sin(2\theta) = 2\sin\theta\cos\theta$, hence we get

$2\sin(2\theta + \frac{\pi}{2})$

We integrate to get $f(\theta)$

$\int 2\sin(2\theta + \frac{\pi}{2}) =- \cos(2\theta + \frac{\pi}{2}) + C$

Hence, we equate the answer to 0 to find θ.

$-\cos(2(0) + \frac{\pi}{2}) + C = 1$

$-\cos(\frac{\pi}{2}) + C = 1$

$C = 1$

Thus the answer is

$-\cos(2\theta + \frac{\pi}{2}) + 1$

2

Finding areas of graphs using integrals and definite integrals

Areas of graphs using integrals can be found by defining the upper and lower boundaries, and then adding them.

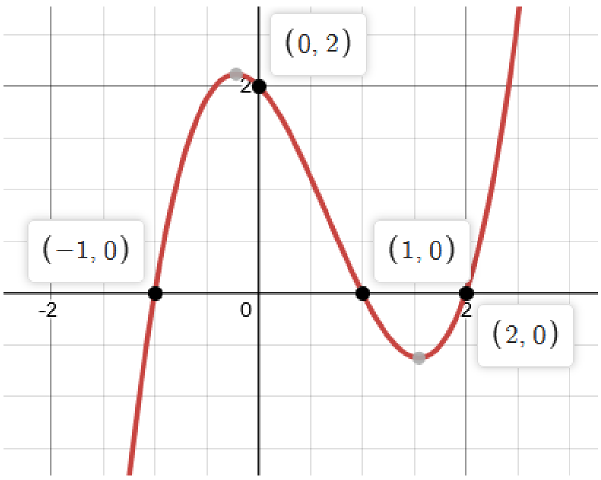
$$\int_a^b |f'(x)| = [|f(x)|]_b^a$$
$$= |f(a)| - |f(b)|, \text{ where } a < b$$

Example (Example 8, Page 458):

a) Factorise the expression $2 - x - 2x^2 + x^3$
b) Hence sketch the graph $f(x) = 2 - x - 2x^2 + x^3$
c) Find the area of the region bounded by the graph $f(x) = 2 - x - 2x^2 + x^3$ and the x axis.

A.
$(1 - x)(1 + x)(2 - x)$

B.
Roots at $x = 1, -1, 2$
y intercept at $2, 0$
Because the coefficient of x^3 is positive, that means it follows the S shape where the left side begins from negative y numbers



C.

$$\int_{-1}^{1} 2 - x - 2x^2 + x^3 + \left|\int_{1}^{2} 2 - x - 2x^2 + x^3\right|$$

$$\left[2x - \frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4}\right]_{-1}^{1} + \left|\left[2x - \frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4}\right]_{1}^{2}\right|$$

LHS

$$\left[2x - \frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4}\right]_{-1}^{1}$$

$$= \left(2(1) - \frac{(1)^2}{2} - \frac{2(1)^3}{3} + \frac{(1)^4}{4}\right) - \left(2(-1) - \frac{(-1)^2}{2} - \frac{2(-1)^3}{3} + \frac{(-1)^4}{4}\right)$$

$$= \frac{13}{12} - \left(-\frac{19}{12}\right)$$

$$= \frac{8}{3}$$

RHS

$$\left|\left[2x - \frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4}\right]_{1}^{2}\right|$$

$$= \left(2(2) - \frac{(2)^2}{2} - \frac{2(2)^3}{3} + \frac{(2)^4}{4}\right) - \left(2(1) - \frac{(1)^2}{2} - \frac{2(1)^3}{3} + \frac{(1)^4}{4}\right)$$

$$= \frac{2}{3} - \frac{13}{12} = -\frac{5}{12}$$

$\left|-\frac{5}{12}\right| = \frac{5}{12}$ because, obviously, the area of something cannot be negative as we're calculating the total area.

Thus,

$$\frac{8}{3} + \frac{5}{12} = \frac{37}{12}$$

3

Exponents and Logarithms

General rules of logarithms

$\log_a xy = \log_a x + \log_a y$(SL 1.7)

$\log_a \frac{x}{y} = \log_a x - \log_a y$(SL 1.7)

$\log_a x^m = m\log_a x$(SL 1.7)

$\log_a x = \frac{\log_b x}{\log_b a}$(SL 1.7) where $b \in R$, you choose any number

$a^x = b$, then $x = \log_a b$(SL 1.5)

Example 1 (Example 13, Page 463):

The value of a sailing boat depreciates at a rate of 15% per year for the first three years. After that, the rate $r\%$ of depreciation stays constant. A new boat costing $60,000 is worth one fifth of its original value after 15 years. Find:

a. The value of the boat, to the nearest dollar, after three years

b. The rate of depreciation after the three years of purchase.

A.

$60000 \times 0.85^3 = \$36848$

B.

$36848 \times n^{12} = 12000$

$n = \sqrt[12]{\frac{12000}{36848}} = 0.9107$

Hence the rate of depreciation is $\simeq 8.93\%$

Example 2 (Example 11, Page 462):

Solve $2^x - 2^{2-x} = 3$

$2^x - 2^2 \times 2^{-x} = 3$

$2^x - \frac{2^2}{2^x} = 3$

$(2^x)^2 - 4 = 3 \times 2^x$

$(2^x)^2 - 3(2^x) - 4 = 0$

Let $\alpha = 2^x$

$\alpha^2 - 3a - 4 = 0$

$(\alpha + 1)(\alpha - 4) = 0$

Powers cannot make a number equal negative, thus the only solution is when $\alpha = 4$

$2^x = 4$

$x = 2$

Example 4 (Example 21, Page 469):

Solve the following questions

a. $\log_{15}x + \log_{15}(2x - 1) = 1,\ x > 0$

b. $\log_4 x + \log_4(x - 6) = 2,\ x > 0$

A.

$\log_{15}(x(2x - 1)) = 1$

$\log_{15}(x(2x - 1)) = \log_{15}15$

Divide both sides by $\log_{15}$

$2x^2 - x = 15$

$2x^2 - x - 15 = 0$

$(2x + 5)(x - 3)$

$x = 3$,since we cannot have a negative answer

<u>OR</u>

$\log_{15}(x(2x - 1)) = 1$

Putting in regular form

$15^1 = 2x^2 - x$

$0 = 2x^2 - x - 15$

$(2x + 5)(x - 3)$

$\therefore x = 3$,since we cannot have a negative answer

B.

$\log_4(x(x - 6)) = 2$

$\log_4(x(x - 6)) = 2\log_4 4$, because 1=$\log_a a$

$\log_4(x(x - 6)) = \log_4 16$

Divide by $\log_4$

$x^2 - 6x - 16 = 0$

$(x + 2)(x - 8)$

$\therefore x = 8$, since we cannot have a negative answer

<u>OR</u>

$\log_4(x(x - 6)) = 2$

Putting into regular form we get

$4^2 = x^2 - 6x$

$0 = x^2 - 6x - 16$

$(x + 2)(x - 8)$

$\therefore x = 8$, since we cannot have a negative answer.

Example 5 (Example 23, Page 471):

Calculate the number of terms that are required for the sum of the geometric series given by

$$\sum_{i=1}^{n} 3 \times 2^i > 1000$$

$r = 2$
$u_1 = 6$

Sum of geometric series is given by

$$S_n = \frac{6(2^n - 1)}{2-1} > 1000$$

$$S_n = 2^n > \frac{1000}{6} - 1$$

$n > \log_2(\frac{1000}{6} - 1)$

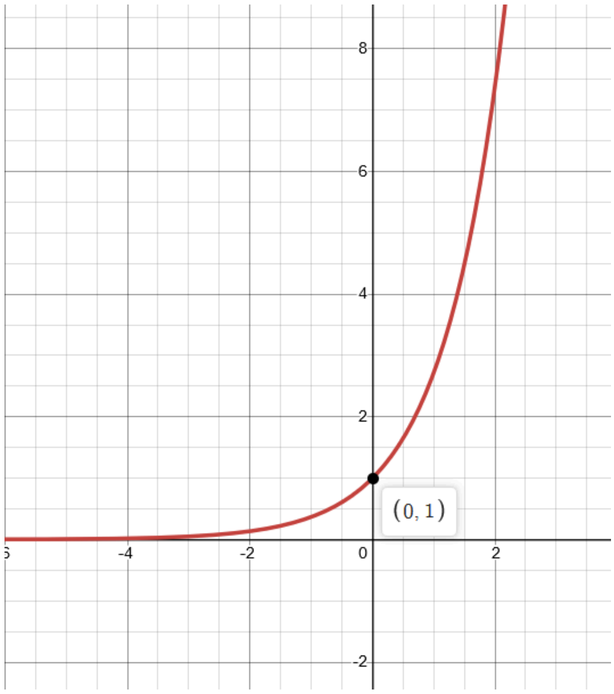
4

Exponential Graphs

Important: If the graph has an asymptote at the x axis, then it is an exponential graph. If the graph has an asymptote at the y axis, then it is a logarithmic graph.

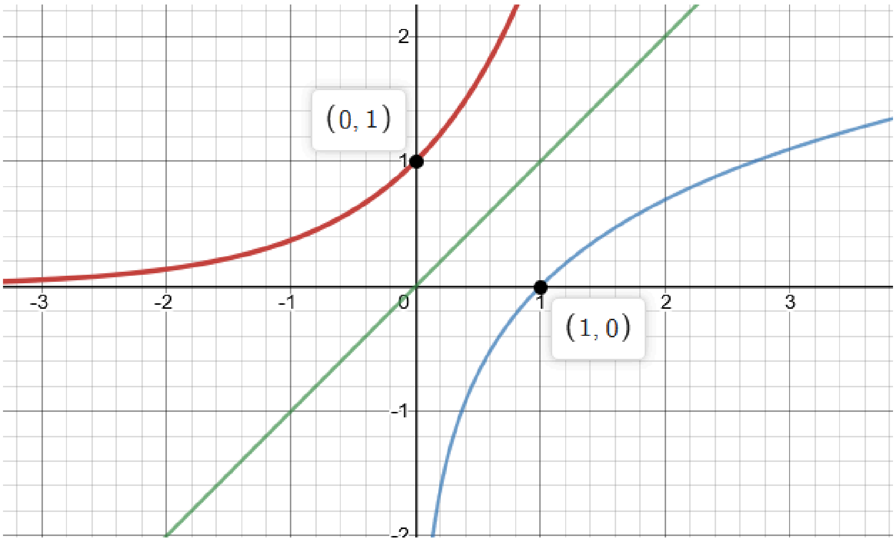
Exponential and logarithmic graphs have special properties.

$y = e^x$graph has an asymptote at at $y = 0$ because the power cannot transform the e into a 0 or negative numbers.



If the x turns into a $- x$, then the graph follows a translation of a reflection in the y axis. Furthermore, if the coefficient of $x$is < 1,then it could be expressed in $- x$ power form, as such the graph is also a reflection of an a^xgraph.

Logarithmic graphs are exponential graphs that undergo a reflection over the $y = x$ function.



The graph shown above expresses $y = e^x$ and $y = \log_e x$ otherwise as $\ln x$. Logarithmic graphs have an asymptote at $x = 0$ because, if thought about it, x is the answer of a question $x = e^y$ and x cannot be negative or 0 because powers cannot transform the result into a negative or 0.

5

<u>Integration by inspection</u>

Integration by inspection is a process of first deriving the integral and seeing the difference it creates if you were to integrate it.

This is mainly done by reverse chain rule, where

$$\frac{d}{dx}(f(g(x))) = f'(g(x)) \times g'(x) \Rightarrow \int f'(g(x)) \times g'(x)\, dx = f(g(x)) +$$

(Not in the formula booklet)

Example 1 (Example 34, Page 492):

Find the following integrals by inspection.

$\int \sin \frac{\pi x}{2} dx$

We let the functions be defined as

$f(u) = \cos(u),\ g(x) = \frac{\pi x}{2}$

Using reverse chain rule we get

$f'(g(x)) \times g'(x) = -\frac{2}{\pi} \times - \cos(\frac{\pi x}{2}) \times \frac{\pi}{2} = \sin(\frac{\pi x}{2})$

(This method is really confusing and inefficient when working out in paper, this can be otherwise solved as knowing that when you derive $\sin x$ or $\cos x$, we multiply by the derivative of the inside. We can do it the opposite, integrate the $\sin x$ or $\cos x$ and afterwards divided by the derivative inside.)

<u>Example 2 (Example 36, Page 493):</u>

Integrate

$\int \sin x \cos^4 x\, dx$

We let the initial function be defined as

$f(x) = \cos^5 x = (\cos x)^5$

Deriving we get

$f'(x) = 5 \times - \sin x \times (\cos x)^4 = - 5 \sin x \cos^4 x$

Thus by deriving we get and know that the difference is $-\frac{1}{5}$, hence we add it to the original $f(x)$., thus the answer is

$-\frac{1}{5}\cos^5 x$

6

Integration by substitution

Integration by inspection is a process of first deriving the integral and seeing the difference it creates if you were to integrate it.

When the integrand contains a quadratic radical expression in the denominator, use one of the following trigonometric substitutions to transform the integral.

- If the form is $\frac{1}{\sqrt{a^2 - x^2}}$ use the substitution $x = a \sin u$.
- If the form is $\frac{1}{\sqrt{x^2 - a^2}}$ use the substitution $x = a \sec u$.
- If the form is $\frac{1}{\sqrt{a^2 + x^2}}$ use the substitution $x = a \tan u$.

Example 1 (Example 39, Page 497):

Find the following integrals using appropriate substitution:

a. $\int 3(6x - 1)e^{3x^2 - x}\, dx$

b. $\int \cot x\, dx$

A.

$\int 3(6x - 1)e^{3x^2 - x}\, dx$

Let $u = 3x^2 - x$, then
$\frac{du}{dx} = 6x - 1$, otherwise as $du = (6x - 1)dx$, substituting in we get

$3\int e^u du = 3e^u$

Substituting the u back in we get

$3e^{3x^2 - x}$

B.

$\int \cot x \, dx = \int \frac{\cos x}{\sin x} dx$

Let $u = \sin x$, then
$\frac{du}{dx} = \cos x$, otherwise as $du = \cos x \, dx$, substituting in we get

$\int \frac{1}{u} du = \ln|u| + C$

Substituting the u back in we get

$\ln|\sin x| + C$

Example 2 (Example 42, Page 499):

Find $\int 3^x dx$.

Knowing that $e^{\ln k} = k$, we can rewrite the equation as $\int e^{\ln 3^x}$

$\int e^{x\ln 3}$

Let $u = x \ln 3$, then $\frac{du}{\ln 3} = dx$

Substituting back in we get

$\frac{1}{\ln 3} \int e^u du$

$\frac{1}{\ln 3} \times e^u + C$

Substituting u back in we get

$\frac{1}{\ln 3} \times e^{\ln 3^x} + C$

$\frac{1}{\ln 3} \times 3^x + C$

$$\frac{3^x}{\ln 3} + C$$

7

Integration by parts

Integration by parts is integrating by choosing one part u and other part by $\frac{dv}{du}$, and differentiating the u whilst integrating $\frac{dv}{du}$. You should always choose u as a part that is not repeated, i.e x but not e^x or $\sin x$ for example, because these functions will repeat. It is also the product rule but in integral form.

$uv - \int v \frac{du}{dx} dx$(AHL 5.16)

Example 1 (Example 45, Page 506):

Find the integral $\int \ln x \, dx$

We first rewrite as

$\int \ln x \times 1 \, dx$, hence

Let $u = \ln x$, $\frac{dv}{dx} = 1$, then

$\frac{du}{dx} = \frac{1}{x}$, $v = x$

As such, applying the integration by parts rule, we get:

$x \ln x - \int x \frac{1}{x} dx$

Integrating we get

$x(\ln x - 1) + C$

Example 2 (Example 46, Page 506):

Find the integral $\int \arccos x \, dx$

We first rewrite as

$\int \arccos\ x \times 1\, dx$

Let $u = \arccos\ x,\ \frac{dv}{dx} = 1$, then
$\frac{du}{dx} = \frac{1}{\sqrt{1-x^2},}\ v = x$
As such, applying the integration parts rule, we get:

$x \arccos\ x\ - \int \frac{x}{\sqrt{1-x^2}} dx$

Let $u = 1 - x^2$, then
$\frac{du}{-2} = x\, dx$

Substituting we get

$x \arccos\ x - \frac{1}{2} \int u^{-\frac{1}{2}} du$

$x \arccos\ x - \frac{1}{2}(2u^{\frac{1}{2}})$

$x \arccos\ x - \frac{1}{2}(2(1 - x^2)^{\frac{1}{2}})$

$x \arccos\ x - \sqrt{1 - x^2} + C$

Example 3 (Exercise 7J, 5, Page 507):

Find $\int (\frac{1+2x}{3}) \sec^2 \frac{x}{2}\ dx$

Let $u = \frac{1+2x}{3},\ \frac{dv}{dx} = \sec^2 \frac{x}{2}$, then
$\frac{du}{dx} = \frac{2}{3},\ v = 2 \tan \frac{x}{2}$

As such, applying the integration by parts rule, we get:

$\frac{1+2x}{3} \times 2 \tan \frac{x}{2} - \int \frac{2}{3} \times 2 \tan \frac{x}{2} dx$

$\frac{1+2x}{3} \times 2 \tan \frac{x}{2} - \int \frac{4}{3} \tan \frac{x}{2} dx$

$\frac{1+2x}{3} \times 2 \tan \frac{x}{2} - \frac{4}{3} \int \frac{\sin \frac{x}{2}}{\cos \frac{x}{2}} dx$

Let $u = \cos\frac{x}{2}$, then

$\frac{du}{dx} =- \frac{1}{2}\sin\frac{x}{2}$

$- 2du = \sin\frac{x}{2}dx$

Substituting back in we get

$\frac{1+2x}{3} \times 2\tan\frac{x}{2} + \frac{8}{3}\int\frac{1}{u}du$

$\frac{1+2x}{3} \times 2\tan\frac{x}{2} + \frac{8}{3}\ln|u|$

$\frac{1+2x}{3} \times 2\tan\frac{x}{2} + \frac{8}{3}\ln(\cos\frac{x}{2}) + C$

Example 4 (Exercise 7J, 12, Page 507):

Find $\int \log_a x$

Using change of base formula we get

$\log_a x = \frac{\ln x}{\ln a}$

Hence

$\int\frac{\ln x}{\ln a} = \frac{1}{\ln a}\int \ln x$

$\frac{1}{\ln a}\int 1 \times \ln x$

Let $u = \ln x$, $\frac{dv}{dx} = 1$, then

$\frac{du}{dx} = \frac{1}{x}$, $v = x$

As such, applying the integration by parts rule, we get:

$x\ln x - \int\frac{x}{x}dx$

$x\ln x - x + C = x(\ln x - 1) + C$

As such, applying this integral in the original equation we get:

$\frac{x(\ln x-1)}{\ln a} + C$

Example 5 (Exercise 7J, 7, Page 507):

Find $\int \frac{x}{3^x}\, dx$

Let $u = x$, $\frac{dv}{dx} = 3^{-x}$, then

$\frac{du}{dx} = 1$, $v = \int 3^{-x}$

To find v, we let $u =- x$, then $\frac{du}{dx} =- 1$, hence

$- \int 3^{u} du$, then

$- \frac{3^{u}}{\ln 3}$, going back to the original equation and applying integration by parts we get:

$- \frac{3^{-x}}{\ln 3} \times x - \int - \frac{3^{-x}}{\ln 3}$

$- \frac{3^{-x}}{\ln 3} \times x + \frac{1}{\ln 3} \int 3^{-x}$, using our previous answer we get

$- \frac{3^{-x}}{\ln 3} \times x + \frac{1}{\ln 3} (- \frac{3^{-x}}{\ln 3})$

$- \frac{3^{-x}}{\ln 3} (x + \frac{1}{\ln 3}) + C$

8

Cyclic integration

Cyclic integration happens when the function keeps repeating itself when integrated by parts. This happens because it is defined by parts such as e^x, $\sin x$ etc.

Example (Exercise 7L, 1, Page 509):

Find $\int \tan x \sec^2 x \, dx$

We first equate $I = \int \tan x \sec^2 x \, dx$

Let $u = \tan x$, $\frac{dv}{dx} = \sec^2 x$, then
$\frac{du}{dx} = \sec^2 x$, $v = \tan x$

$I = \tan^2 x - \int \tan x \sec^2 x \, dx$, hence we substitute in I, thus

$I = \tan^2 x - I$
$I = \frac{tan^2 x}{2}$

8

BONUS

Hyperbolic substitution

Knowing the following:

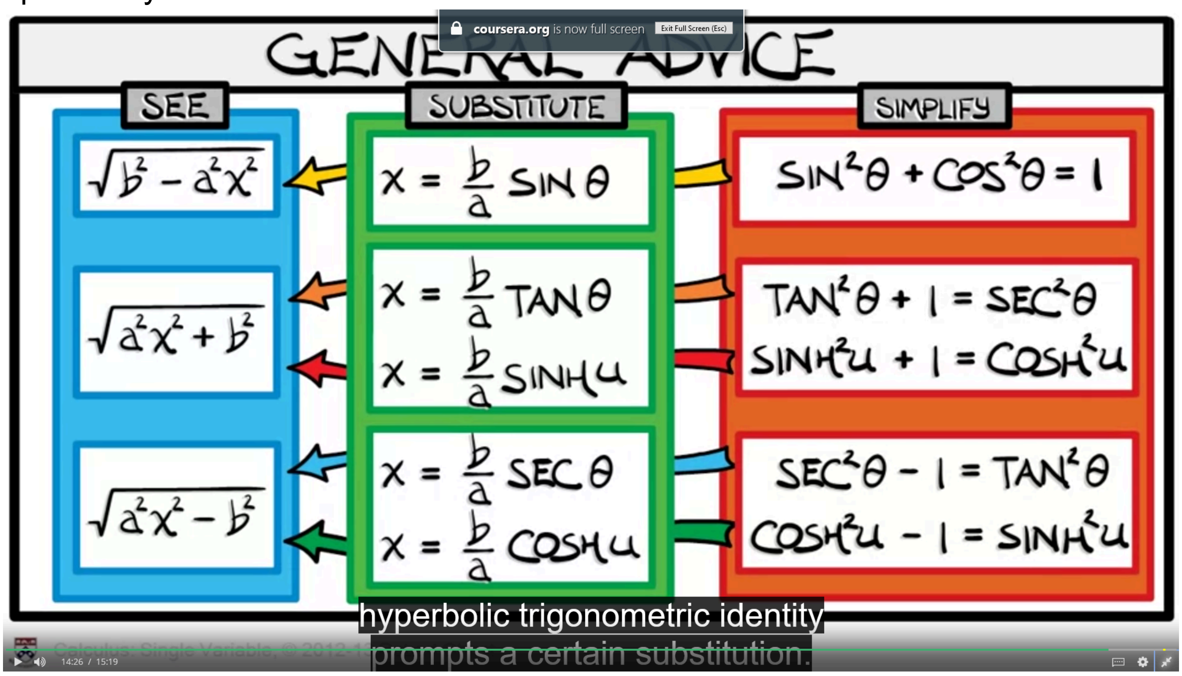
$$\sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} = \cosh x = \frac{e^{x}+e^{-x}}{2}$$

$$\sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!} = \sinh x = \frac{e^{x}-e^{-x}}{2}$$

Then, the identity follows true:

$$\cosh^{2}(x) - \sinh^{2}(x) = 1$$

This can be exploited in substitution.
Specifically



Example 1 (UPenn Course):

$$\int \frac{dx}{\sqrt{x^2 - 6x + 10}}$$

Sol 1, using tan sub
Completing the square to get

$$\int \frac{1}{\sqrt{(x-3)^2+1}} dx$$

Using tan sub
$x - 3 = \tan u$
$dx = \sec^2 u \, du$

$$\int \frac{\sec^2 u}{\sqrt{\sec^2 u}} du$$

$$\int \sec u \, du = \ln|\sec u + \tan u|$$

$$= \ln\left|x - 3 + \sqrt{x^2 - 6x + 10}\right| + C$$

Sol 2, using sinh sub
Completing the square to get

$$\int \frac{1}{\sqrt{(x-3)^2+1}} dx$$

Using sinh sub
$x - 3 = \sinh u$
$dx = \cosh u \, du$

Substituting

$$\int \frac{\cosh u}{\sqrt{\sinh^2 x + 1}} du$$

Using identity $\cosh^2 x - \sinh^2 x = 1$

$$\int \frac{\cosh u}{\cosh u} du$$

$= u + C$
$= arcsinh(x - 3) + C$

Generalisation (use only when the original form of tan has a square root):
Identity $\cosh^2 x - \sinh^2 x = 1$
Use $\sinh$ sub if in the form, where $x = a \sinh u$

$\frac{1}{\sqrt{x^2+a^2}}$

Use $\cosh$sub if in the form, where $x = a\cosh u$

$\frac{1}{\sqrt{x^2-a^2}}$

And remember that $\frac{d}{dx}\sinh x = \cosh x$ and $\frac{d}{dx}\cosh x = \sinh x$, where signs do not alternate.

Topic 8 Advanced Calculus

1

Finding area between 2 graphs

Finding the area between graphs is simple and intuitive. One needs to refine the new y axis that the person wants to find the area. This is done by taking the upper graph and then subtracting it from the bottom graph with defined limits.

Example 1 (UPenn Course):

Find the area inscribed between $\sqrt{x}$ and x^2

Let us first find the intersection points, one of which is obviously 0.

$\sqrt{x} = x^2$

$x = x^4$

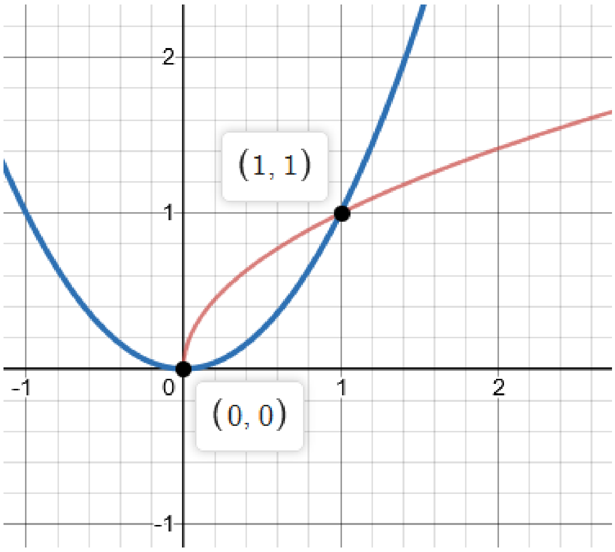
$1 = x^3$

$x = 1$

Hence, the limits are 0 and 1.

Knowing that $\sqrt{x}$ approaches infinity faster in the interval of 0 and 1 we can deduce that it is on top. (e.g. $0.1^2 = 0.01$ whilst $\sqrt{0.01} = 0.1$. One gets smaller, other gets bigger for such intervals).

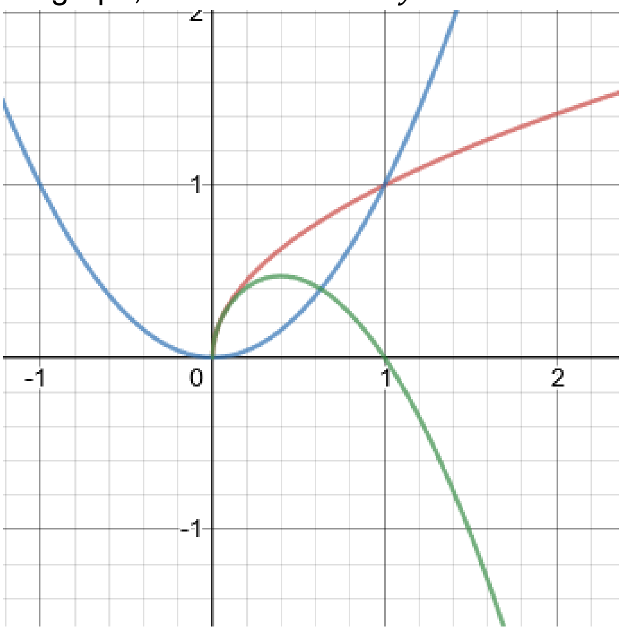
A visual graph for aid:



In order to find the area, we must minus the non-relevant bit (which is the bottom part of x^2 , hence

$\sqrt{x} - x^2$

This will give us a new graph, with a new set of y axis as a result of subtraction:



Hence, the area will be

$\int_0^1 (\sqrt{x} - x^2)dx$

Evaluating to get

$\frac{1}{3}$

2

Volumes and Volumes of revolution

Volumes of revolution is a method of having a function and then rotating it about 360 degrees in order to obtain a 3D volume of an object.

- The volume of a solid of revolution formed when $y = f(x)$, which is continuous in the interval $[a, b]$, is rotated 2π radians about the x-axis is $V = \pi\int_a^b y^2 \mathrm{d}x$.
- The volume of a solid of revolution formed when $y = f(x)$, which is continuous in the interval $y = c$ to $y = d$, is rotated 2π radians about the y-axis is $V = \pi\int_c^d x^2 \mathrm{d}y$.

- If $f(x) \geq g(x)$ for all x in the interval $[a, b]$, then the volume of revolution formed when rotating the region between the two curves 2π radians about the x-axis is $V = \pi\int_a^b \left([f(x)]^2 - [g(x)]^2\right)\mathrm{d}x$.
- If x_1 and x_2 are relations in y such that $x_1 \geq x_2$ for all y in the interval $[c, d]$, then the volume formed by rotating the region between the two curves 2π radians about the y-axis is $V = \pi\int_c^d \left(x_1^2 - x_2^2\right)\mathrm{d}y$.

This works because of the formula πr^2, in which r is replaced with the function, depending whether it is about the x or the y axis. This allows to evaluate an r which is the area between given intervals.

Example 1 (Example 6, Page 526):

Find the volume of the solid formed when the region when the region between the two curves $f(x) = \sqrt{\frac{x}{2}}$ and $g(x) = \frac{x^2}{4}$ is rotated 2π about

a. The x axis
b. The y axis

A.

One should note that $x = 0$ is an obvious solution to $f(x) = g(x)$

Moreover, equating the two to obtain our other intersection

$$\sqrt{\frac{x}{2}} = \frac{x^2}{4}$$

$$\frac{x}{2} = \frac{x^4}{16}$$

$8 = x^3$
$x = 2$

Hence, we can then deduce that

$$\int dV = 2\pi \int_0^2 (\sqrt{\frac{x}{2}})^2 - (\frac{x^2}{4})^2 \quad dx$$
$$V = \left[\frac{x^2}{4} - \frac{x^5}{80}\right]_0^2$$
$$V = \frac{3}{5}\pi \; cubic\; units$$

B.

For the $y$axis, we will have to make x the subject in each.
$2y^2 = x$
$2\sqrt{y} = x$

To calculate the boundaries we just put our x boundaries and solve for y
$2y^2 = 0, \; y = 0$
$2y^2 = 2, \; y = 1$

We know that when inspected through the y axis we know that $\frac{x^2}{2} > \sqrt{\frac{x}{2}}$
Hence

$$\int dV = \pi \int_0^1 (2\sqrt{y})^2 - (2y^2)^2 dy$$
$$V = \pi\left[2y^2 - \frac{4}{5}y^5\right]_0^1$$
$$V = \frac{6\pi}{5} \; cubic\; units$$

Example 2 (Example 5, Page 526):

Find the volume enclosed by the region between the graphs of $y = \frac{x^2+1}{2}$ and the line $y = 3$, by 2π radians.

We know that the upper limit is 3. Let us find the minima so we can determine the lower boundary.
$\frac{dy}{dx} = x = 0$, so minima has $(0, y)$

Plugging in to find y
$y = 0.5$

Thus, knowing the formula for surfaces of revolution and making x the subject to obtain
$\sqrt{2y - 1}$
$\int dV = \pi \int_{0.5}^{3} (\sqrt{2y - 1})^2$
$= 6.25\pi\ cubic\ units$

3

Taylor & Maclaurin Series

Taylor and Maclaurin series is a way of expressing functions as a polynomial. It is good for approximation. The higher the degree found, the better approximation derived.

When discovering Maclaurin series, Maclaurin realised the following:

In the first iteration, this polynomial, let's call it $P(x) = \sin x$, is only known to share the same y-intercept, but after the second iteration, it has the same gradient there too, and after the third, it has the same "rate of change of gradient" or acceleration, and then after blah blah and so on and so forth.

It is possible to compute 2 different Taylor/Maclaurin series and then multiply/apply them together if a question is in the appropriate form. See example 1.

Proof of Maclaurin

Let an infinite polynomial $f(x)$ be defined as the following:

$$f(x) = a + bx + cx^2 + dx^3 + ex^4 + ...$$

where $a, b, c, d, e \in R$ and denote the coefficient of the n degree of the polynomial.

Notice that, if we substitute $x = 0$, then we can find a.

$$f(0) = a$$

Let us differentiate the function.

$$f'(x) = b + 2(cx) + 3(dx^2) + 4(ex^3) + ...$$

Notice that, if we substitute $x = 0$, then we can find b.

$$f'(0) = b$$

Let us differentiate $f'(x)$.

$$f''(x) = 2c + 3 \times 2(dx) + 4 \times 3(ex^2) + ...$$

Notice that, if we substitute $x = 0$, then we can find c.

$$f''(0) = 2c, \ \frac{f''(0)}{2} = c$$

Let us differentiate $f''(x)$.

$$f'''(x) = 3 \times 2(d) + 4 \times 3 \times 2(ex) + ...$$

Notice that, if we substitute $x = 0$, then we can find d.

$$f'''(0) = 3 \times 2(d),\ \frac{f'''(0)}{3!} = d$$

Let us differentiate $f'''(x)$.

$$f^4(x) = 4 \times 3 \times 2(e) + ...$$

Notice that, if we substitute $x = 0$, then we can find e.

$$f^4(0) = 4 \times 3 \times 2(e),\ \frac{f^4(0)}{4!} = e$$

And the list continues. Now substitute our derived definitions of a, b, c, d, e into $f(x)$ to obtain

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + ... + \frac{f^n(0)}{n!}x^n$$

Where $n \in N$and lists in ascending order.

$f(x) = f(0) + x f'(0) + \frac{x^2}{2!}f''(0) + ...$ (AHL 5.19)

However, do carefully notice that this allows for a good approximation at $x = 0$. What if $x = a$?

Then we can apply translations to our generalisation.

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f'''(a)}{3!}(x - a)^3 + ... + \frac{f^n(a)}{n!}(x - a)^n$$

This is the Taylor series.

This is not required for the syllabus, but it is good to know. (I wish I had a series named after me at $x = 1$)

Example 1 (from a Coursera UPenn course):

Find the Maclaurin expansion of $e^{1-\cos x}$ up to the 4th degree.

First find the Maclaurin of $1 - \cos x$ to the 4th degree.

$f(0) = 0$
$f'(x) = \sin x,\ f'(0) = 0$
$f''(x) = \cos x,\ f''(0) = 1$
$f'''(x) =- \sin x,\ f'''(0) = 0$

$f^4(x) =- \cos x,\ f^4(0) =- 1$

Therefore

$e^{\frac{1}{2!}x^2 - \frac{1}{4!}x^4 + \ldots}$

Computing Maclaurin for e^x

$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}\ldots$

Substituting in for x from our previously computed Maclaurin

$$= 1 + (\frac{1}{2!}x^2 - \frac{1}{4!}x^4) + \frac{(\frac{1}{2!}x^2 - \frac{1}{4!}x^4)^2}{2!}$$
$$= 1 + (\frac{1}{2!}x^2 - \frac{1}{4!}x^4) + \frac{\frac{1}{4}x^4}{2}$$
$$= 1 + \frac{1}{2}x^2 - \frac{1}{24}x^4 + \frac{3}{24}x^4$$
$$= 1 + \frac{1}{2}x^2 + \frac{1}{12}x^4$$

---

and then combining them.

Worked example 4.7

Using the Maclaurin series for $\sin x$ and e^x, find the series expansion of $e^{\sin x}$ as far as the term in x^4.

We start by substituting the series for $\sin x$, only going as far as the x^4 term

$$e^{\sin x} = e^{x - \frac{x^3}{3!} + \ldots}$$
$$\approx e^x e^{-\frac{x^3}{3!}}$$

We now use the series for e^x only going as far as x^4 and then expand

$$\approx \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right)\left(1 + \left(-\frac{x^3}{3!}\right) + \cdots\right)$$
$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^3}{6} - \frac{x^4}{6} + \cdots$$
$$\approx 1 + x + \frac{x^2}{2} - \frac{x^4}{8}$$

Otherwise, it is possible, for example, to split the function into 2 and then find the Maclaurin/Taylor for both and then multiply each other.

4

Limits using Taylor

Limits can also be done using Taylor. This is especially useful for equations which are cyclic in l'Hopital, i.e. differentiating does not help.

For example (from UPenn course):
(this can also be solved by l'Hopital, but is much more tedious)

$$\lim_{x\to 0} \frac{e^{x^2}-1}{1-\cos x}$$

Computing taylor series for each

$$\cos x = 1 - \frac{x^2}{2!} + \ldots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots$$

$$e^{x^2} = 1 + x^2 + \frac{x^4}{4}$$

Substituting back in

$$\lim_{x\to 0} \frac{(1+x^2+\frac{x^4}{4}+\ldots)-1}{1-(1-\frac{x^2}{2!}+\ldots)}$$

$$\lim_{x\to 0} \frac{x^2+\frac{x^4}{4}+\ldots}{\frac{x^2}{2!}+\ldots}$$

$$\lim_{x\to 0} \frac{x^2(1+\frac{x^2}{4}+\ldots)}{x^2(\frac{1}{2}+\ldots)}$$

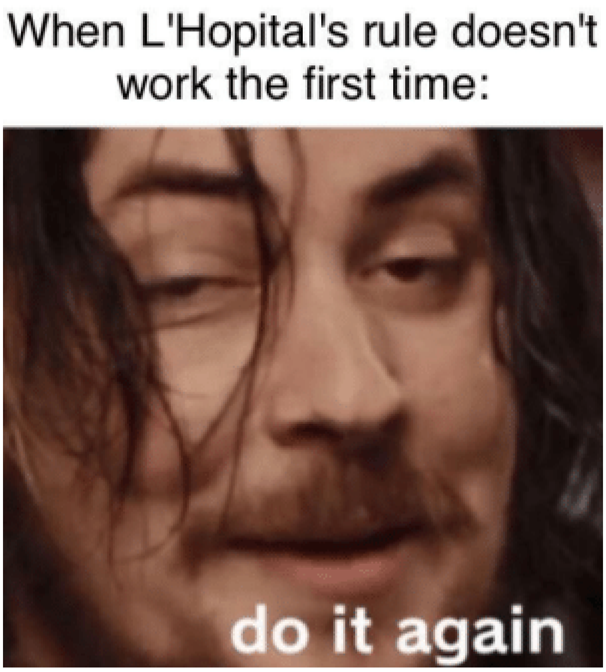
$$\lim_{x\to 0} \frac{1}{\frac{1}{2}} = 2$$

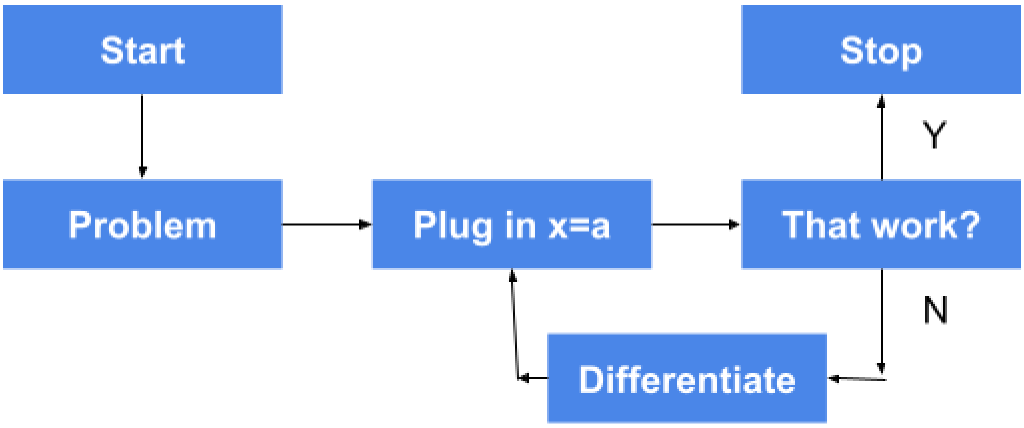
5

I'Hopital's Rule

I'Hopital states that the limit of a continuous function can be found by differentiating if given in the form

$\lim_{x \to a} \frac{f(x)}{g(x)}$, where $\lim_{x \to a} \frac{0}{0}$ or $\lim_{x \to a} \frac{\infty}{\infty}$, then one can obtain the limit by differentiating continuously i.e. $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ until such that $f(x)$ or $g(x)$ is now applicable in limits..





If it is cyclic (no matter how many times it is differentiated, will always obtain 0/0), then it is better to evaluate using Taylor.

Example **(From a UPenn Course)**:

Evaluate $\lim_{x \to 1} \frac{\cos(\frac{x\pi}{2})}{\sqrt{x}-1}$

Consider $f(x) = \cos(\frac{x\pi}{2})$, then $f'(x) =- \frac{\pi}{2}\sin(\frac{x\pi}{2})$
At $f'(1) =- \frac{\pi}{2}$

Consider $g(x) = \sqrt{x} - 1$, then $g'(x) = \frac{1}{2}\frac{1}{\sqrt{x}}$
At $g'(x) = \frac{1}{2}$.

Hence
$\lim_{x \to 1} \frac{-\frac{\pi}{2}}{\frac{1}{2}} =- \pi$

6

Separable Autonomous Ordinary Differential Equations

Ordinary differential equations are used to model specific types of things. It is important to get differentials to their unknowns respectively in order to solve.

Separable Autonomous Ordinary Differential Equations are equations which follow the form

$$\frac{dx}{dt} = f(x)$$

Example 1 (UPenn Course):

4. On a cold day you want to brew a nice hot cup of tea. You pour boiling water (at a temperature of $212^{o}F$) into a mug and drop a tea bag in it. The water cools down in contact with the cold air according to Newton's law of cooling:

$$\frac{dT}{dt} = \kappa(A - T)$$

where T is the temperature of the water, $A = 32^{o}F$ the ambient temperature, and $\kappa = 0.36$ min^{-1}.

The threshold for human beings to feel pain when entering in contact with something hot is around $107^{o}F$. How many seconds do you have to wait until you can safely take a sip? Round your answer to the nearest integer.

Let us rearrange the equation such that

$$\frac{dT}{A-T} = kdt$$

Integrating

$$\int \frac{dT}{A-T} = \int kdt$$

$$-\ln(A - T) = kt + C$$

Putting everything to the power of e

$$\frac{1}{A-T} = e^{kt+C}$$

$\frac{1}{A-T} = e^{kt} \times e^{C}$, since e^{C} is the initial condition, or rather, the y intercept we can denote it as y_0

$$\frac{1}{A-T} = y_o e^{kt}$$

Knowing that initial temperature is $212^{\circ}F$ we can substitute for T
We also know $A = 32$ and since this is initial condition, $t = 0$

$$\frac{1}{32-212} = y_0$$
$$y = -\frac{1}{180}$$

Now solving for $T = 107$, $A = 32$, and $k = 0.36$
$$\frac{1}{32-107} = -\frac{1}{180}e^{0.36t}$$
$$\frac{12}{5} = e^{0.36t}$$
$$\ln\frac{12}{5} = 0.36t$$
$$\frac{\ln\frac{12}{5}}{0.36} = t$$
$$t = 2.43\ min$$
$$t = 146\ s$$

7

Integrating factor

Non-autonomous differential equations are such that $\frac{dy}{dx} = P(x)y + Q(x)$ and require an integrating factor for solution.

If the $a\frac{dy}{dx}$ has coefficient, then divide everything by that coefficient a.

PROOF:

Let
$$\frac{dy}{dx} = P(x)y + Q(x)$$

Let us denote the integrating factor as I. Multiplying the equation by I to obtain
$$I\frac{dy}{dx} = IP(x)y + IQ(x)$$

Let us rearrange such that
$$I\frac{dy}{dx} - IP(x)y = IQ(x)$$

Recall the product rule of differentiation
$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

Notice that the first part of the differential equation forms the RHS of the product rule. As a result we then can write that, to obtain a product rule form of the equation,
$$-I \times P(x) = \frac{dI}{dx}$$

(which would as a result give $I\frac{dy}{dx} + \frac{dI}{dx}y = IQ(x)$which is equivalent to $\frac{d}{dx}(Iy) = IQ(x)$)

So now, we can solve the obtained differential equation, in which
$$-P(x)dx = \frac{dI}{I}$$

$$-\int P(x)dx = \int \frac{dI}{I}$$

$$-\int P(x)dx = \ln I$$

$e^{-\int P(x)dx} = I$

Substituting back into or original equation gives us the differential of the product rule of I and x

$\frac{d}{dx}(I \times y) = IQ(x)$

Integrating so we can solve for x

$\int \frac{d}{dx}(I \times y) = \int IQ(x)$

$e^{-\int P(x)dx} \times y + C = \int Q(x)e^{-\int P(x)dx}$

$y = e^{\int P(x)dx} \int Q(x)e^{-\int P(x)dx} + C$

Or

$y = \frac{1}{I}(\int Q(x)I\, dx)$

Integrating factor for $y' + P(x)y = Q(x)$

$$e^{\int P(x)dx} \quad \text{(AHL 5.18)}$$

Example 1 (UPenn Course):

Which of the following is the *integrating factor* used to solve the following linear differential equation?

$$t^2\frac{dx}{dt} = 4t - t^5x$$

Putting it into standard form, and dividing everything by t

$t\frac{dx}{dt} + t^4x = 4$

Putting everything to the integrating factor and factorising t (MAKE SURE TO ISOLATE DY/DX or DX/DT etc.)

$(I\frac{dx}{dt} + It^3x)t = 4$

To obtain the product rule inside the bracket we set

$It^3 = \frac{dI}{dt}$

$\int t^3 dt = \int \frac{dI}{I}$

$\frac{t^4}{4} + C = \ln I$

$I = e^{\frac{t^4}{4}+C}$

Example 2 (UPenn Course):

Solve the differential equation $\frac{dx}{dt} - 5x = 3$.

Multiplying everything by I

$I\frac{dx}{dt} - 5Ix = 3$

Then we know, in order to obtain the product rule on LHS we must

$-5I = \frac{dI}{dt}$

$\int -5dt = \int \frac{dI}{I}$

$-5t + C = \ln I$

$I = e^{-5t+C} = A_0 e^{-5t}$

Since we have obtained the product rule then

$\int d(I \times x) = \int 3I$

$I \times x = \int 3A_0 e^{-5t} dt$

$I \times x = -\frac{3}{5}A_o e^{-5t} + C$

$x = \frac{-\frac{3}{5}A_0 e^{-5t} + C}{A_0 e^{-5t}}$

$x = -\frac{3}{5} + Ce^{5t}$

Example 3 (UPenn Course):

1. Solve the differential equation $\frac{dx}{dt} = \frac{x}{1+t} + 2$.

Multiplying everything by I

$I\frac{dx}{dt} = \frac{Ix}{1+t} + 2I$

$I\frac{dx}{dt} - \frac{Ix}{1+t} = 2I$

To obtain a product rule we know that

$-\frac{I}{1+t} = \frac{dI}{dt}$

$-\int\frac{dt}{1+t} = \int\frac{dI}{I}$

$-\ln(1+t) + C = \ln I$

$\frac{A_0}{1+t} = I$

Having obtained the product rule we get that

$\int d(I \times x) = \int 2I$

$I \times x + C = 2A_0\int\frac{1}{1+t}dt$

$\frac{A_0}{1+t} \times x + C = A_0(\ln 1 + t) + C$

$\frac{A_0}{1+t} \times x = A_0\ln(1+t) + C$

$x = \ln(1+t)(1+t) + C(1+t)$ the $\frac{C}{A_0}$ is still a constant C

8

Homogeneous O.D.E

9

<u>Euler's Method</u>

Euler's method is computing the estimation of an O.D.E in the form
$\frac{dy}{dx} = f(x, y)$
Because calculus will not help solve such a differential equation. In order for such thing to be estimated, an initial condition $x(y_0) = x_0$ must be given.

Our goal is to approximate
$x_{\circ} = x(y_{\circ})$
$y = y_n = y_0, y_1, y_2, \ldots, y_N$
$x = x_n = x_0, x_1, x_2, \ldots, x_N$
$y_N = y_{\circ}$
$x_N = x_{\circ}$
Step size h denotes the amount of steps that one does before calculating the next term. The smaller the step size the more accurate the result.

Euler's Method for solving differential equations numerically:

Given a first order differential equation in the form $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$:

- select the initial point (x_0, y_0)
- use the recurrence formulas $x_{n+1} = x_n + h$ and $y_{n+1} = y_n + hf(x_n, y_n)$ to generate as many points as instructed
- plot the points (x_n, y_n) and connect with a smooth curve.

$y_{n+1} = y_n + h \times f(x_n, y_n); x_{n+1} = x_n + h$, where h is a constant (step length)
(AHL 5.18)

Example 1 (Page 549, Exercise 8I, Q3):

Use Euler's method with a step size of 0.1 to find an approximate value of y when $x = 0.4$ that satisfies the differential equation $y' = x^2 + y^2$ with the initial condition $y(0) = 1$. Explain whether your approximate value is greater than or less than the actual value.

We must first create a table for this.

n	x_n	y_n	$\frac{dy}{dx}$ computed at n

0	0	1	1
1	0.1	$y_1 = 1 + 0.1 \times 1 = 1.1$ We use our formula to compute	$1.1^2 + 0.1^2 = 1.22$ We substitute x_1 and y_1 to obtain this
2	0.2	$y_2 = 1.1 + 0.1 \times 1.22 = 1.222$	$1.222^2 + 0.2^2 = 1.533284$
3	0.3	$y_3 = 1.222 + 0.1 \times 1.533284$ $= 1.3753284$	$1.3735^2 + 0.3^2 = 1.98152820$
4	0.4	$y_4 = 1.3735 + 0.1 \times 1.98152820$ $= 1.573481221$	

Final ans, to 3 s.f. is $y \approx 1.57$

Approximate is less because our output is increasing each time, getting closer to the real value.

10

BONUS

All Taylor series can be expressed as infinity sums

E.g.

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x$$

$$\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = \cos x$$

$$\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = \sin x$$

$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}\ |x| < 1$ (sum of a geometric)

$$\sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1} = \ln(1 - x)|x| < 1$$

$$\sum_{k=0}^{\infty} (-1)^k \frac{x^{k+1}}{k+1} = \ln(1 + x)|x| < 1$$

$$\sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} = \cosh x = \frac{e^x + e^{-x}}{2}$$

$$\sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!} = \sinh x = \frac{e^x - e^{-x}}{2}$$

Proof for geometric series formula

Let $S_n = 1 + x + x^2 + x^3 + ...$

Multiplying everything by x we obtain
$xS_n = x + x^2 + x^3 + x^3 + ...$

Minusing from S_n
$S_n - xS_n = 1$

Factoring

$S_n(1 - x) = 1$

$S_n = \frac{1}{1-x}$

Hence, notice that the first term always stays, thus $1 = u_1$.
Finally, notice that x is the ratio, which means $r = x$. This holds true as long as $|x| < 1$ for convergence.

Thus

$S_n = \frac{u_1}{1-r} |x| < 1$

Given that this is a geometric to infinity, then it can be expressed in infinity sum form.

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \; |x| < 1$$

Proof that $\sum_{k=0}^{\infty} (-1)^k \frac{x^{k+1}}{k+1} = \ln(1 + x) |x| < 1$

First, notice that $\frac{d}{dx}\ln(1 + x) = \frac{1}{1+x}$, hence $\int \frac{1}{1+x} = \ln 1 + x$

Moreover, notice that $\frac{1}{1+x}$ is the same as a geometric series with $- x$.

Hence, it can be expressed as $\frac{1}{1+x} = \sum_{k=0}^{\infty} (- x)^k$ However, remember that we have to integrate, thus

$\int \sum_{k=0}^{\infty} (- x)^k$ that is, $\sum_{k=0}^{\infty} \int (- 1)^k x^{\ \ k}$

$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} x^{k+1} + C$

Since when $x = 0$ $\ln 1 = 0$, along when $x = 0$ in our sum we have to obtain 0 which holds true if $C = 0$.

Finally,

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} x^{k+1} = \ln(1 + x) \; |x| < 1$$

Topic 9 Vectors

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Topic 10 Advanced Complex Numbers

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Topic 11 Sets & Distributions

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Revision sources

IB Discord Revision papers made by Andrew:
https://drive.google.com/drive/folders/1QgQwYq72LH5tFrcyUMQ7loE-ZKdVmXGS

Fake mocks by Andrew:
https://drive.google.com/drive/folders/1XqzLyQMSGuJTRqFf06R3ByE4wrZi6UlA

Revision Village question bank & more:
https://www.revisionvillage.com/ib-math-analysis-and-approaches-hl/

IB Math HL videos:
https://www.youtube.com/playlist?list=PLUQ9_xf9jKTc8Tt2ADixwp53rYWA8ZeOT
(covers pretty much every topic)

IB Math HL past papers:
https://freeexampapers.com/exam-papers/IB/Maths/Higher/

EXTRA

Source: Part 1 course, Week 3, Challenge Question 2.

1. Suppose that a function $f(x)$ is *reasonable*, so that it has a Taylor series

1 point

$$f(x) = c_0 + c_1 x + c_2 x^2 + \text{H.O.T.}$$

with $c_0 \neq 0$. Then the reciprocal function $g(x) = 1/f(x)$ is defined at $x = 0$ and is also reasonable. Let

$$g(x) = b_0 + b_1 x + b_2 x^2 + \text{H.O.T.}$$

be its Taylor series. Because $f(x)g(x) = 1$, we have

$$(c_0 + c_1 x + c_2 x^2 + \text{H.O.T.})\,(b_0 + b_1 x + b_2 x^2 + \text{H.O.T.}) = 1 + 0x + 0x^2 + \text{H.O.T.}$$

Multiplying out the two series on the left hand side and combining like terms, we obtain

$$c_0 b_0 + (c_0 b_1 + c_1 b_0)x + (c_0 b_2 + c_1 b_1 + c_2 b_0)x^2 + \text{H.O.T.} = 1 + 0x + 0x^2 + \text{H.O.T.}$$

Equating the coefficients of each power of x on both sides of this expression, we arrive at the (infinite!) system of equations

$$\begin{gathered} c_0 b_0 = 1 \\ c_0 b_1 + c_1 b_0 = 0 \\ c_0 b_2 + c_1 b_1 + c_2 b_0 = 0 \\ \cdots \end{gathered}$$

relating the coefficients of the Taylor series of $f(x)$ to those of the Taylor series of $g(x)$. For example, the first equation yields $b_0 = 1/c_0$, while the second gives $b_1 = -c_1 b_0/c_0 = -c_1/c_0^2$.

Using the above reasoning for $f(x) = \cos x$, determine the Taylor series of $g(x) = \sec x$ up to terms of degree two.

The Power Tower

Find $\frac{d}{dx} x^{x^{x^{x^{x^{x^{x^{x}}}}}}}$

Let $y = x^{x^{x^{x^{x^{x}}}}}$, then notice we rewrite as $y = x^y$

Taking ln to remove the exponential

$\ln y = y \ln x$

Implicitly differentiating $d(\ln y = y \ln x)$

$u = y,\ v = \ln x$

$u = dy,\ v = \frac{dx}{x}$

$\frac{dy}{y} = \frac{y}{x}dx + dy\ln x$

$dy(\frac{1}{y} - \ln x) = \frac{y}{x}dx$

$\frac{dy}{dx} = \frac{\frac{y}{x}}{\frac{1}{y} - \ln x}$

$\frac{dy}{dx} = \frac{y^2}{x - yx\ln x}$

$\frac{dy}{dx} = \frac{y^2}{x(1 - y\ln x)}$

<u>Newton Raphson's Linearisation Method</u>

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

OR

$f(x_n) + f'(x_n)(a - x)$

Where x_n is the guessing number and a is the original number

For example,

Estimate $\sqrt{15}$

We estimate that the answer is 4. Hence, $x_n = 4$

$f(x) = x^2 - a$

$f'(x) = 2x$

$x_{n+1} = 4 - \frac{16-15}{8}$

$x_{n+1} = 4 - \frac{1}{8} \simeq 3.875$

In calculator, we obtain that $\sqrt{15} = 3.8729$, which is correct to 3 decimal points.

You are in charge of designing packaging materials for your company's new product. The marketing department tells you that you must put them in a cube-shaped box. The engineering department says that you will need a box with a volume of $500\ \text{cm}^3$. What are the dimensions of the cubical box? Starting with a guess of $8\ \text{cm}$ for the length of the side of the cube, what approximation does one iteration of Newton's method give you? Round your answer to two decimal places.

Hence, $\sqrt[3]{500}$or, $\sqrt[3]{a}$

$f(x) = x^3 + a$

$f'(x) = 3x^2$

$x_{n+1} = 8 - \frac{8^3 - 500}{3 \times 8^2}$

$x_{n+1} = 8 - \frac{1}{16}$

$\simeq 7.9375$

Whilst in calculator we obtain that $\sqrt[3]{500} = 7.9370$

Prove that $\frac{d}{dx}a^x = a^x \ln x$

First, $\frac{d}{dx}(a^x)$

Let $y = a^x$

Putting everything to ln operator

$\ln y = x \ln a$

Implicitly differentiating

$d(\ln y = x \ln a)$

$\frac{dy}{y} = dx \ln a$

Rearranging to get

$\frac{dy}{dx} = y \ln a$

Substituting y back in

$\frac{dy}{dx} = a^x \ln a$

OR

$\frac{d}{dx}e^{x\ln a}$

$= \ln a \times e^{x\ln a}$

$\ln a \times a^x$

Find $\lim_{x\to\infty} (1 + \frac{a}{x})^x$

Let $y = (1 + \frac{a}{x})^x$

Putting everything to ln operator

$\ln y = x \ln(1 + \frac{a}{x})$

Taylor expanding $\ln(1 + \frac{a}{x})$

We know that the first term would be $\frac{a}{x} + O(\frac{1}{x^2})$

Substituting back in

$x(\frac{a}{x} + O(\frac{1}{x^2})$

$= a + O(\frac{1}{x})$

We ignore the O because as $x \to \infty$ it approaches 0.

$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} a$

$\lim_{x\to\infty} (y = e^a)$

$= e^a$

$$\frac{dP(t)}{dt} = bP(t)$$

The rate of growth b is the difference between the birth rate and the mortality rate. The Malthusian model supposes that this rate is constant. Of course we know that does not always happen: factors such as famines, outbreaks of disease or advances in medicine do influence these rates. When modeling some process mathematically it is important to recognize what our assumptions are and when they no longer hold. In this problem and the next we will look at one particular event that would result in a violation of Malthus' assumption that the growth rate b is constant over time: the occurrence of famines.

Experimental data suggests that food production F grows linearly over time:

$$F(t) = F_0 + st$$

We will now make two assumptions:

- Most food is perishable, so that the amount of food available at any given time is exactly the amount produced at that time. This means that we are not taking into account the effect of the possibility of preserving food for long periods of time.
- The amount of food that a person in our population eats is, on average, constant and equal to some number α. That is, the amount of food necessary to keep everybody fed is $\alpha P(t)$.

The so-called *Malthusian catastrophe* happens when there is not enough food to feed the whole population, that is, when $\alpha P(t) = F(t)$.

The Food and Agriculture Organization of the United Nations considers that a person needs around 1800 kcal/day to be considered well-fed. According to a report released in 2002 by the same organization:

- The world population in 2002 was around 6 billion.
- The population growth rate was estimated at 1.1% and expected to remain approximately constant for several decades.
- Total food production in 2002 was determined to be around $6.13 \cdot 10^{15}$ kcal, with an expected growth rate of $1.11 \cdot 10^{14}$ kcal/year.

[Source: FAO, "World Agriculture: Towards 2015/2030. Summary Report", 2002]

Source: Part 3, Week 1, Simple O.D.E challenge homework

Estimate the world population by the year 2030.

- ○ 7.8 billion
- ○ 8.8 billion
- ○ 8.6 billion
- ○ 8.4 billion
- ◉ 8.2 billion
- ○ 8.0 billion

✓ Correct

2. [Continued from previous question] Estimate when the Malthusian catastrophe would happen if our assumptions continue to hold.

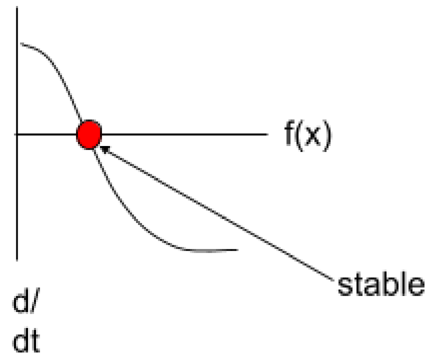
- ○ $39.42e^{1.1t} = 61.3 + 1.11t \Rightarrow t \approx 0.4$ years.
- ◉ $39.42e^{0.011t} = 61.3 + 1.11t \Rightarrow t \approx 166$ years.
- ○ $0.11e^{0.011t} = 61.3 + 1.11t \Rightarrow t \approx 827$ years.
- ○ $0.11e^{1.1t} = 61.3 + 1.11t \Rightarrow t \approx 6$ years.

✓ Correct

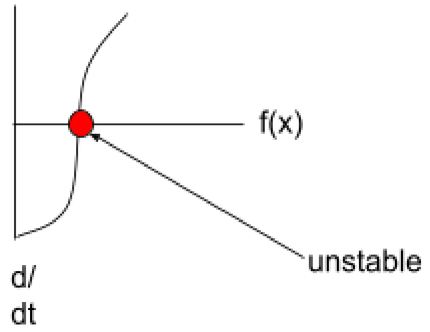
Equilibria points and estimation of differential equations

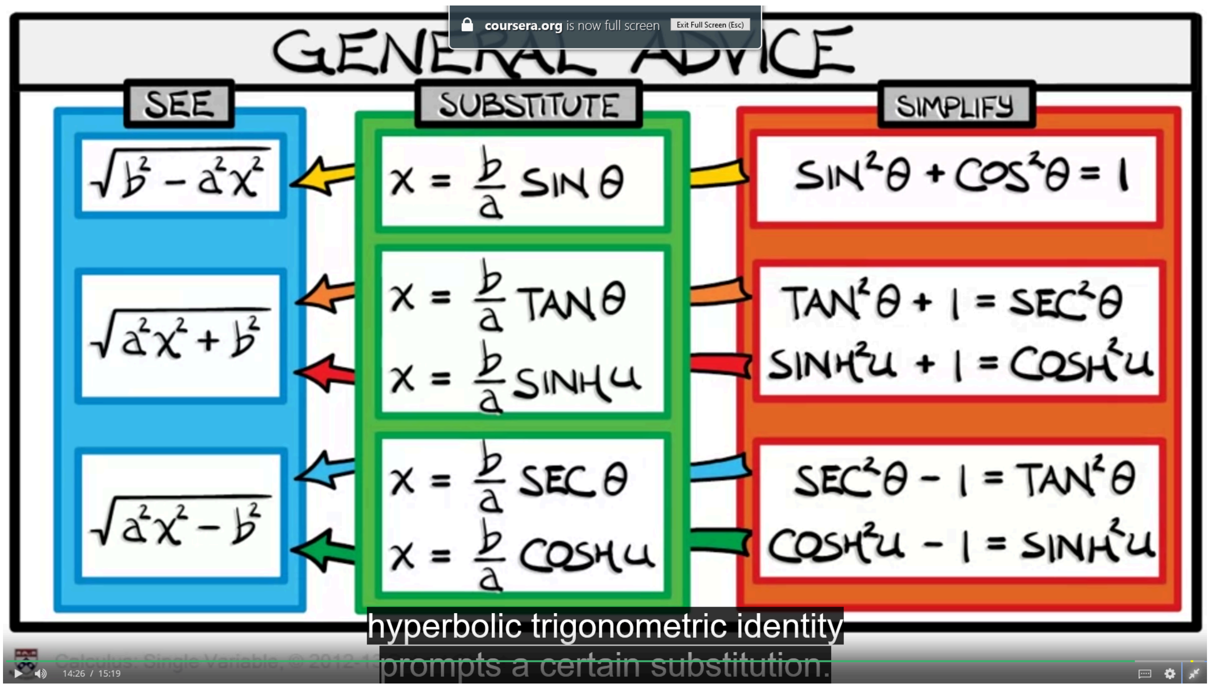
To find equilibria of a differential equation
Let $\frac{d}{dt}$ be denoted as $\hat{x}$
Let
$\hat{x} = f(x)$ or rather $\frac{d}{dt} = f(x)$
Plotting $\hat{x}$ vs $f(x)$ reveals the stability type.
At $f(x) = 0$ there are equilibrium points.

An equilibrium is stable if $f'(equilibrium) < 0$ (decreasing)



An equilibrium is unstable if $f'(equilibrium) > 0$ (increasing)





1. Recall the definition of definite integrals through Riemann sums: 1 point

$$\int_{x=a}^{b} f(x)\,dx = \lim_{P:\Delta x \to 0} \sum_{i=1}^{n} f(x_i)(\Delta x)_i$$

Here P is a partition of the interval $[a, b]$ into n intervals P_i, each of width $(\Delta x)_i$. The point x_i is a sampling of P_i, that is, a point in the interval P_i. The limit is taken over all partitions as the width of the subdivisions gets smaller and smaller.

One particular choice of partition and sampling that can be used to numerically evaluate definite integrals is the following. With n fixed, divide the interval $[a, b]$ into n subintervals P_i of common length $(\Delta x)_i = (b - a)/n$. For the sampling, choose the right endpoint of each P_i; this gives you the formula

$$x_i = a + i\frac{b-a}{n}$$

With these choices of partition and sampling, compute the Riemann sums for the integral

$$\int_{x=1}^{2} \frac{dx}{x}$$

for $n = 1$, $n = 2$ and $n = 3$ subdivisions.

Note: in the next Lecture we will learn that

$$\int_{x=1}^{2} \frac{dx}{x} = \ln 2 \simeq 0.693$$

Compare this value to the ones you obtain from the Riemann sums.

---

2. With the same choices of partition and sampling as in the previous problem, evaluate the Riemann sum for the integral 1 point

$$\int_{x=0}^{3} x^2\,dx$$

for an arbitrary number n of subdivisions.

You might need to use any of the following formulas:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \quad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

Using formulas from above

$$(\Delta x)_i = \frac{3-0}{n}$$

$$x_i = 0 + \frac{3i}{n}$$

$$f(x_i) = x^2$$

Substituting

$$\sum_{i=1}^{n} (\frac{3i}{n})^2 \times \frac{3}{n}$$

$$= \sum_{i=1}^{n} \frac{27i^2}{n^3}$$

Seperating

$$\frac{27}{n^3}\sum_{i=1}^{n} i^2$$

And now substituting what is given

$$\frac{27}{n^3} \times \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{27(2n^3+3n^2+n)}{6n^3}$$

$$= 9 + \frac{27}{2n} + \frac{9}{2n^2}$$

Improper integrals

Improper integrals come in 2 forms

Blow-up and Tail

First it is important to clarify the powers of functions that they converge and diverge to.

If it is given in the form

$$\int_a^{\infty} x^{-p}$$

Then for convergence p must be $p > 1$
For divergence p must be $p \le 1$

If given in the form

$\int_a^{0} x^{-p}$ (0 because it is an asymptote)

Then for convergence p must be $p < 1$
For divergence p must be $p \ge 1$

It is possible to test whether an integral converges or diverges by looking at the leading order term in the MacLaurin series.

Afterwards, replace the limit with any variable and create a limit that approaches the new variable.

---

i. For $0 < a < \infty$:

$$\int_a^{\infty} \frac{1}{x^p}\,dx \begin{cases} \text{converges} & \text{if } p > 1, \\ \text{diverges} & \text{if } p \le 1. \end{cases}$$

ii. For $0 < b < \infty$:

$$\int_0^{b} \frac{1}{x^p}\,dx \begin{cases} \text{converges} & \text{if } p < 1, \\ \text{diverges} & \text{if } p \ge 1. \end{cases}$$

iii. For $-\infty < a < b < \infty$:

$$\left.\begin{aligned} &\int_a^{b} \frac{1}{(x-a)^p}\,dx \\ &\int_a^{b} \frac{1}{(b-x)^p}\,dx \end{aligned}\right\} \begin{cases} \text{converge} & \text{if } p < 1, \\ \text{diverge} & \text{if } p \ge 1. \end{cases}$$

2. Consider the integral

$$\int_{x=0}^{1} \frac{e^{-x}}{x}\,dx$$

It is improper because its integrand blows up at $x = 0$. Using your knowledge of Taylor series, determine the leading order behavior of the integrand near $x = 0$ and decide whether the integral converges or diverges.

Note: observe that you do not need to find an antiderivative in order to determine whether an improper integral converges or diverges!

Maclaurin of e^{-x} is $1 + O(x)$

Hence we obtain $\frac{1}{x}$

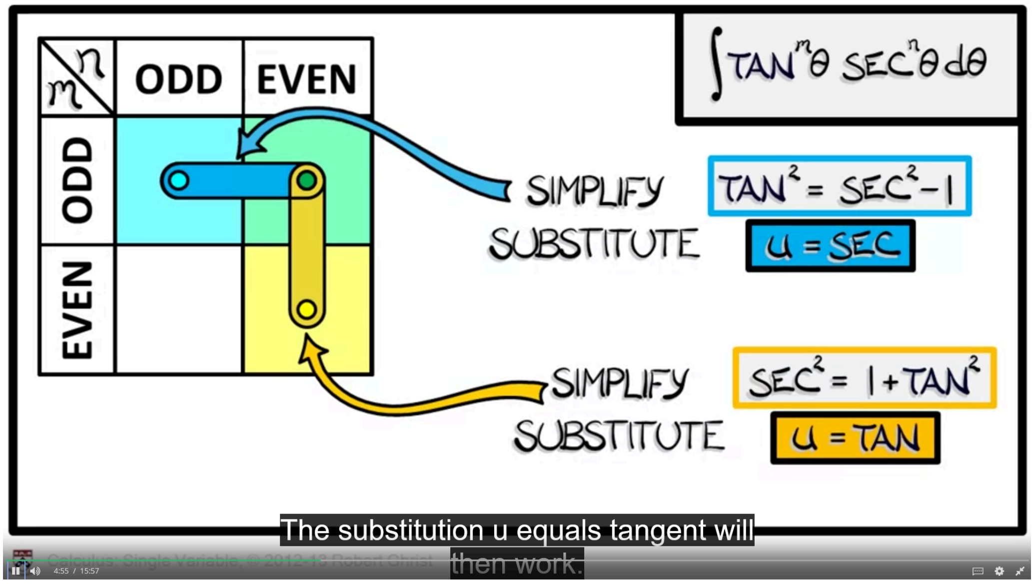
Recalling that at $p = 1$ it is divergent.

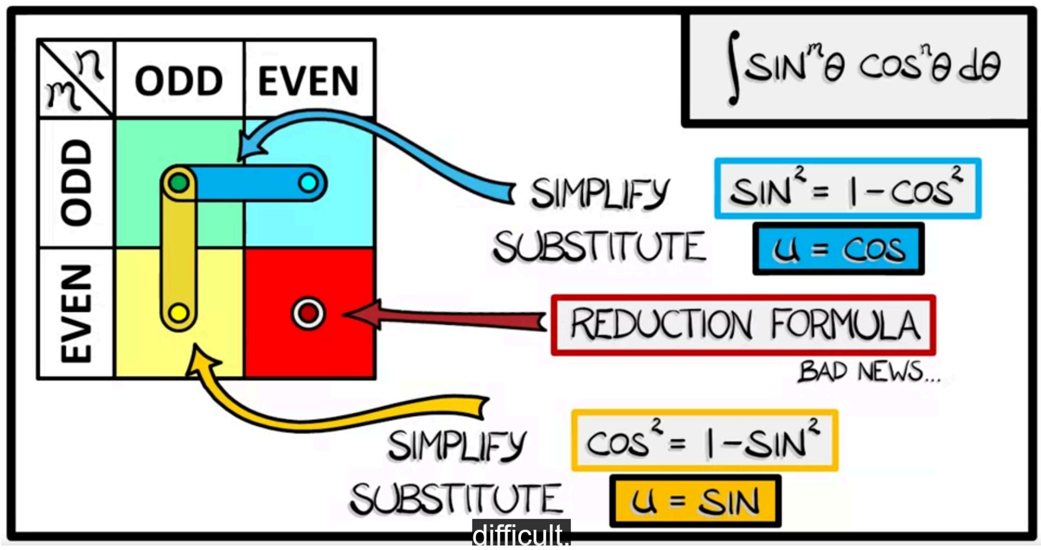
http://www.uop.edu.pk/ocontents/Section4.pdf

Trigonometric integrals

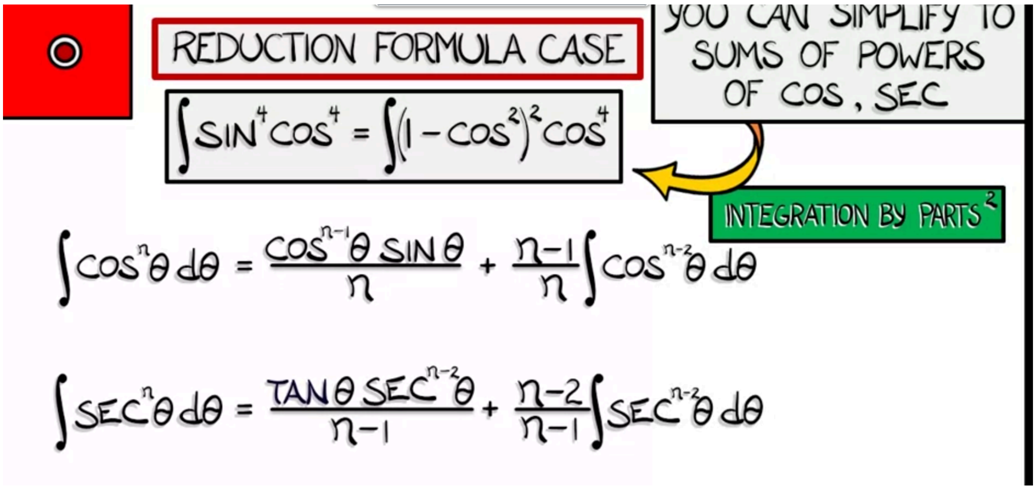
Trigonometric integrals use the means of trigonometric identities in order to work around the integral.

A generalisation is as shown, although it is quite intuitive anyway:





The reduction formula is the following:



3. (a) Use integration by parts to show that

$$\int \sin^n x \, dx = -\frac{\cos x \, \sin^{(n-1)} x}{n} + \frac{(n-1)}{n} \int \sin^{n-2} x \, dx .$$

(b) Use the reduction formula in (a) and the trig. identity $\cos^2 x = 1 - \sin^2 x$ to find

$$\int \sin^2 x \, \cos^2 x \, dx .$$

(This technique is useful when the powers are *even*.)

(c) Use the result in (a) to find $\int_0^{\frac{\pi}{2}} \sin^4 x \, dx$.

Euler's Formula

$e^{ix} = \cos x + i \sin x$

Recall that, through Taylor expansions of $\sin x$ and $\cos x$, they can be written as the following infinite polynomials:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}...$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}...$$

Also, recall the expansion of e^x

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ...$$

Now, substituting ix to x

$$e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + ...$$

Expanding and splitting imaginary with real parts

$$e^{ix} = 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!}...$$
$$e^{ix} = (1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...) + i(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...)$$

And hence, after substitution, we obtain:

$$e^{ix} = \cos x + i \sin x$$

Substituting π to x

$$e^{i\pi} = \cos \pi + i \sin \pi$$
$$e^{i\pi} = - 1 + 0$$
$$e^{i\pi} + 1 = 0$$

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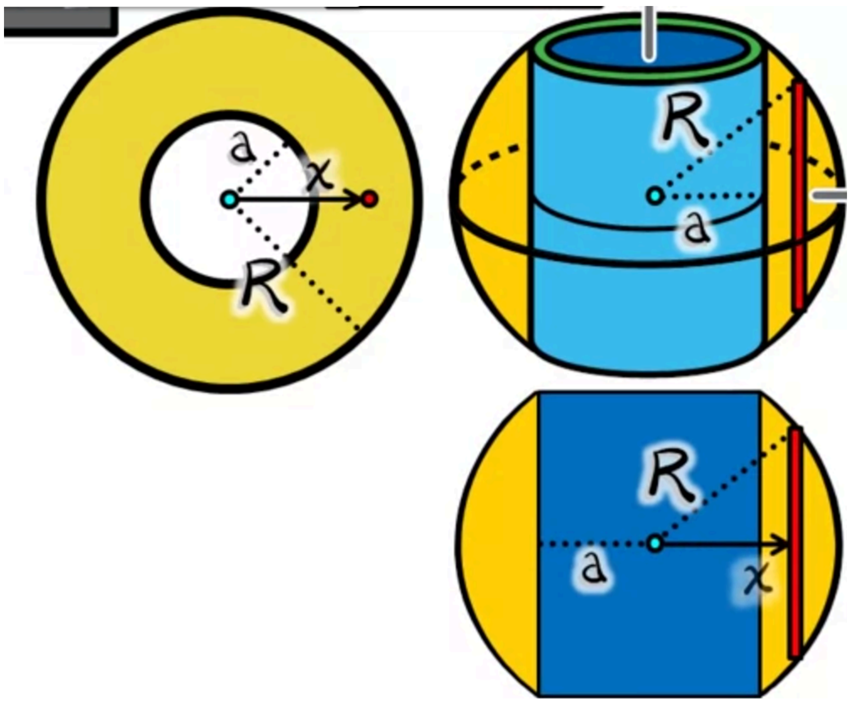
https://www.youtube.com/watch?v=uX0NGCDVfWA

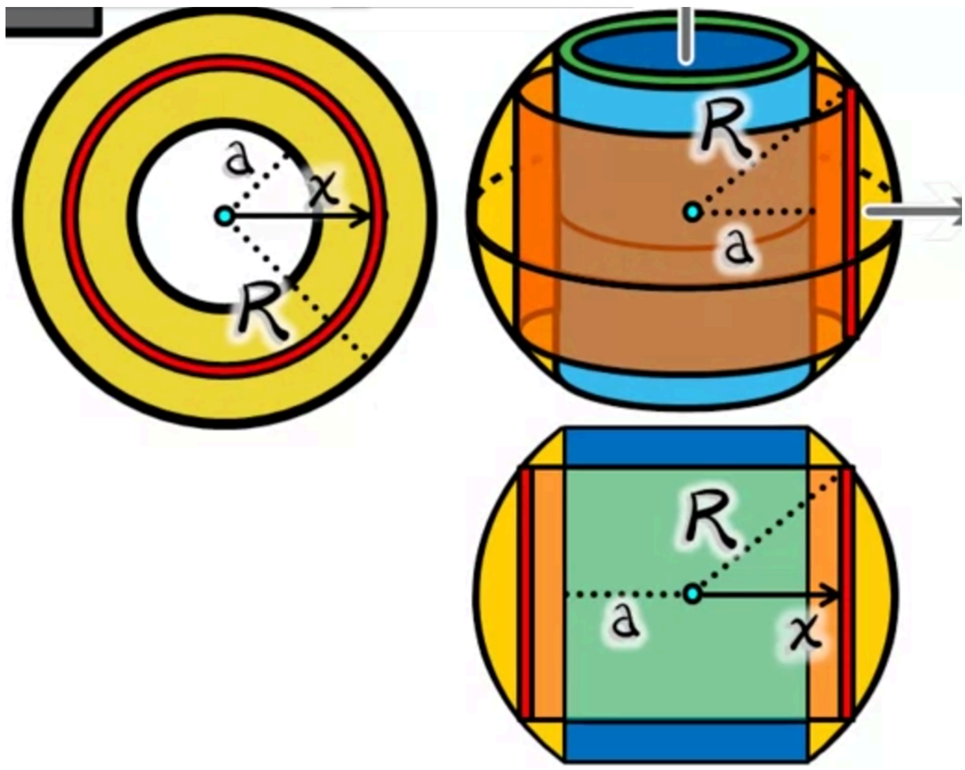
Volumes

Let us calculate the area of a bead.



Consider taking an infinitesimal strip x inside with width dx and height h
The height depends on radius. Then writing the height h using pythagoras where $\frac{h}{2} = \sqrt{R^2 - x^2}$, hence $h = 2\sqrt{R^2 - x^2}$

However now consider expanding the infinitesimal strip x height h width dx to become a circumference. So, $2\pi x$. Giving us a new form:



So, we have obtained such that πx is one of the elements. However, now consider defining the height. So we get we got that height as

$\frac{h}{2} = \sqrt{R^2 - x^2}$, hence $h = 2\sqrt{R^2 - x^2}$

Now consider applying circumference the height to its circumference to obtain

$2\pi x \times 2\sqrt{R^2 - x^2}$

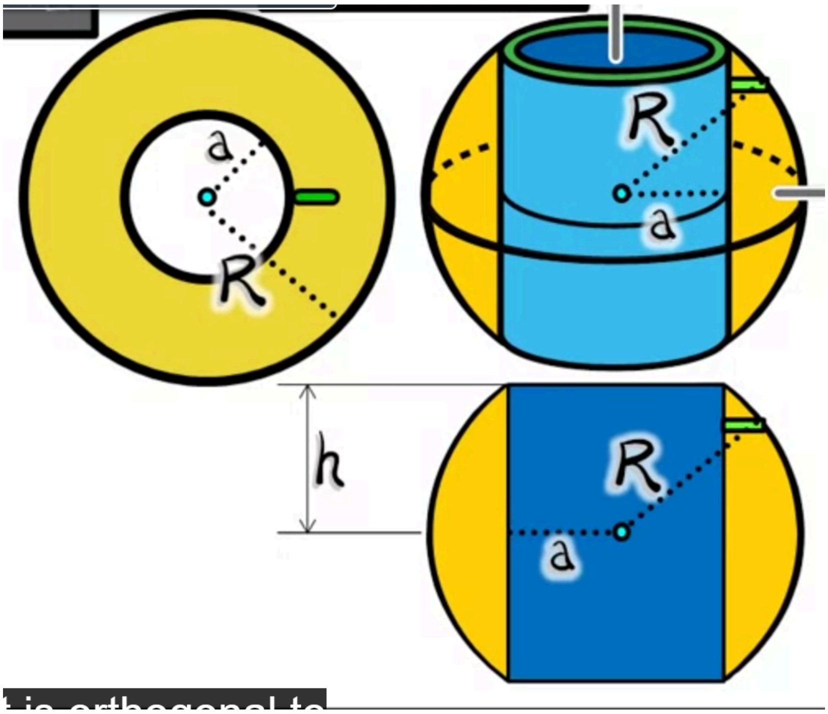
$V = \int dV = \int_a^R 2\pi x \times 2\sqrt{R^2 - x^2}$

Which gives

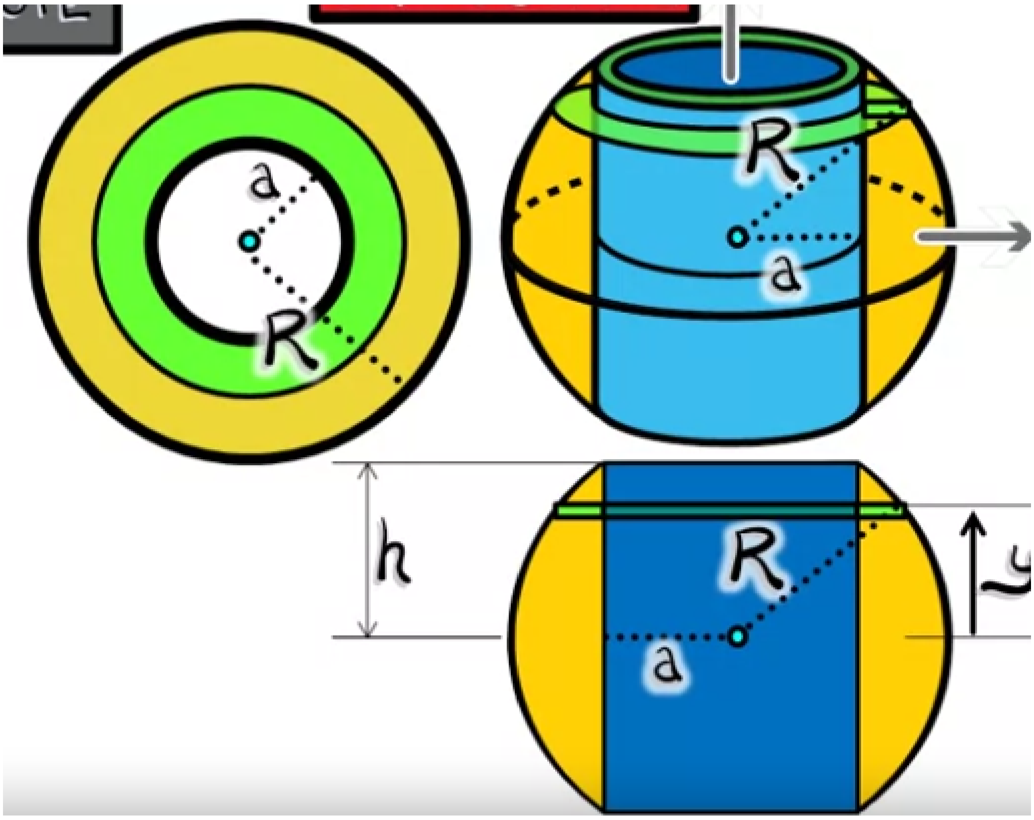
$\frac{4}{3}\pi h^3$

4. Let D be the region under the curve $y = \ln\sqrt{x}$ and above the x-axis from $x = 1$ to $x = e$. Find the volume of the region obtained by revolving D about the x-axis.

Now consider solving orthogonally



Now consider creating a disk out of it



So it is obtained that, through the formula of area of a circle

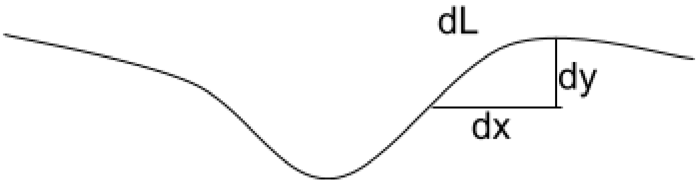
πr^2, which becomes $\pi(\sqrt{R^2 - y^2})^2$
However, this considers the whole shape, so we must subtract the inside of radius a

$\pi(\sqrt{R^2 - y^2})^2 - \pi a^2 = dV$

So $V = \int dV = \int_{-h}^{h} \pi(\sqrt{R^2 - y^2})^2 - \pi a^2$

Arc Length

Consider curve which



Through pythagoras we obtain that
$dL = \sqrt{dx^2 + dy^2}$ We can manipulate it using the chain rule such that
$dL = \sqrt{dx^2 + (\frac{dy}{dx}dx)^2}$
$dL = \sqrt{1 + (\frac{dy}{dx})^2}dx$

Moreover consider a parametric curve which is implicit in nature;

Then knowing that
$dL = \sqrt{dx^2 + dy^2}$
Using the chain rule to obtain
$dL = \sqrt{(\frac{dx}{dt}dt)^2 + (\frac{dy}{dt}dt)^2}$
$dL = \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}dt$

Average

The average considered to be such that the above area in the interval $[a, b]$ is the same as the below area after crossing a line.

$$\bar{f} = \frac{\int_a^b f(x)dx}{\int_a^b dx}$$

If the function is circular, such as $\sin$ then apply the following

$$\bar{f} = \sqrt{\frac{\int_a^b f(x)^2 \mathrm{d}x}{\int_a^b \mathrm{d}x}}$$

Centroids and infinitesimal square area computation

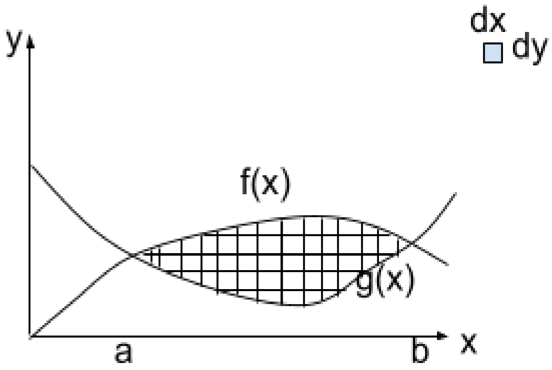
A centroid is the center of a function in domain D. e.g. triangles It is given by the coordinates $(\bar{x}, \bar{y})$

Recall that average is given by

$$\bar{x} = \frac{\int_D x}{\int_D 1}$$

Consider computing squares with infinitesimal lengths, of dy and dx. Hence the area of such a square is $dy\ dx$. I.e. $dA = dy\ dx$

Recall that, in order to find the area between two graphs that



Recall that, to find the area in between one could compute it as

$$\int_a^b f(x) - g(x)dx$$

However, consider finding the area using squares of $dy\,dx$

$$\iint_D dA$$

Given that our infinitesimal area is given by $dy\,dx = dA$

$$\iint dy\,dx$$

The upper limit of such is given by the upper function $f(x)$, whilst the lower is $g(x)$

So, $\int \int_{g(x)}^{f(x)} dy\,dx$

Gives

$\int f(x) - g(x)\,dx$, which is the same as above.

Going back to the original question of centroids, of which

$$\bar{x} = \frac{\int_D x}{\int_D 1}$$

Otherwise

$$\bar{x} = \frac{\iint_D x\,dA}{\iint_D dA}$$

Recalling that $dA = dy\,dx$

$$\bar{x} = \frac{\iint_D x\,dy\,dx}{\iint_D dy\,dx}$$

One should notice that the integral on the denominator is the same as one that we have just computed in order to gain an understanding. Hence, it can be generalised as area A.

$$\overline{x} = \frac{1}{A}\int\int x\,dy\,dx$$

Solving for the first integral, x has nothing to do with y, hence it can be treated like a constant. The boundaries are the same, of which the higher function minus the lower.

$$\overline{x} = \frac{1}{A}\int x\left(\int_{g(x)}^{f(x)} dy\right) dx$$

(These are just your regular brackets, by the way, not some weird function notation)
At the same time, let us substitute the limits for the dx integral

$$\overline{x} = \frac{1}{A}\int_a^b x(f(x) - g(x))\,dx$$

Now let us compute for $\overline{y}$. It is important to note that as we are integrating for dy first, we may not treat y as a constant anymore like we did for x, thus the computation will be different.

$$\overline{y} = \frac{\int_D y}{\int_D 1}$$

$$\overline{y} = \frac{\int\int_D y\,dA}{\int\int_D dA}$$

$$\overline{y} = \frac{\int\int_D y\,dy\,dx}{\int\int_D dy\,dx}$$

$$\overline{y} = \frac{1}{A}\int\int_{g(x)}^{f(x)} y\,dy\,dx$$

$$\overline{y} = \frac{1}{A}\int_a^b \frac{1}{2}\left((f(x))^2 - (g(x))^2\right) dx$$

Center of mass

$$\overline{x} = \frac{\int_D x\,dM}{\int_D dM}$$

$\bar{x} = \frac{1}{M}\int_D x\, dM$

Where dM is an equation of mass, typical given by rho ρ for a one dimensional density of an object

---

Moments and Gyrations

Moments and gyrations are calculated using the formula

$I = r^2 M$

Or otherwise

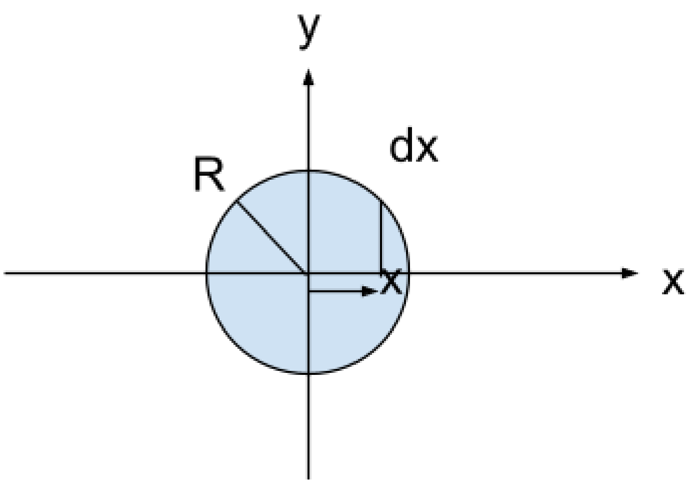
$dI = r^2 dM$

Where

r is the distance from the center of mass

dM is the mass element, which can be denoted as ρ dA where rho is the density. dA varies with the shape respectively.

Consider a disc as an example



Hence the infinitesimal strip dx height can be calculated using pythagoras which yields

$\sqrt{R^2 - x^2} = h$

However consider symmetry so

$2\sqrt{R^2 - x^2}$

Now consider

$dI = r^2 dM$

Decomposing into the following

$dI = r^2 \rho\, dA$

We know $dA = 2\sqrt{R^2 - x^2}\, dx$

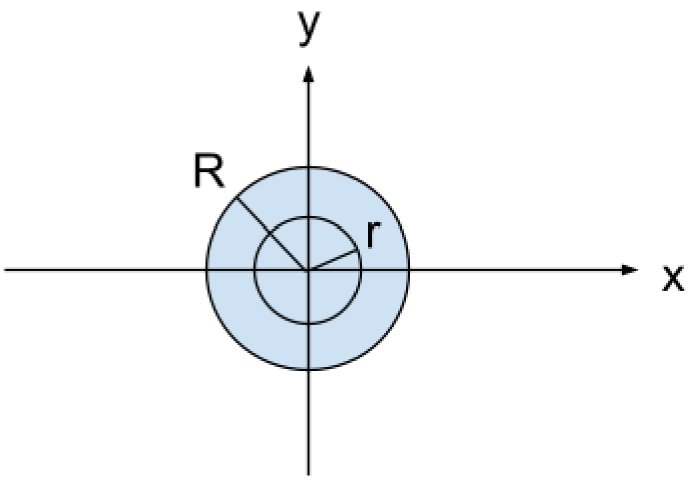
And we know r is x so we obtain

$2 \times 2 \times \rho \int_0^R x^2 \sqrt{R^2 - x^2} dx$

The 2 comes from symmetry. Notice that for limits we begin with 0, and never touch the negative numbers. As a result, we must consider it.

This evaluates into $\frac{1}{2} MR^2$

Another perspective to the problem, considering an infinitesimal circle



$dI = r^2 dM$

Decomposing into

$dI = r^2 \rho\, dA$

Considering dA

Given that $A = \pi r^2$, then $dA = 2\pi r$

$dA = 2\pi r\, dr$

Hence we obtain

$dI = r^2 \times 2 \times \pi \times r \times \rho\, dr$

$I = 2\pi\rho \int_0^R r^3$

$= \frac{1}{2} MR^2$ knowing that $\pi r^2 \rho = M$

Discrete Differences

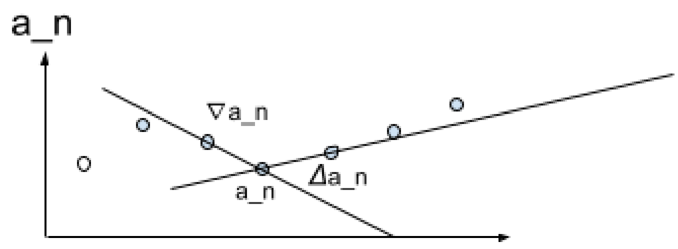
For discrete calculus. We can differentiate but in a different way. Consider the two following definitions

$$f'(x) = \lim_{h \to 0} \left(\frac{f(x+h)-f(x)}{h}\right) \ \& \ \Delta a_n = \frac{a_{n+1}-a_n}{1}$$

One can observe that the difference is small, and that they are very similar. But we have h defined strictly as 1 for the discrete case.

Now consider the backward difference.
$\nabla a_n = \frac{a_n - a_{n-1}}{1}$ (The symbol is called a 'Nabla' respectively)

If one was to plot the discrete values, then you would realise you find the difference between 2 points, similar to finding the gradient.



For example, consider $4n$
$4n = 0, 4, 8, 12, 16, 20, 24, ...$
$\Delta 4n = 4, 4, 4, 4, 4, 4, ...$
It is a linear, constant.

Consider Fibonacci.
$F = 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ...$
$\Delta F = 1, 0, 1, 1, 2, 3, 5, 8, 13, 21, ...$
One obtains a Fibonacci but with a shift.

Consider the sequence 2^n
$2^n = 1, 2, 4, 8, 16, 32, 64, 128, 256, ...$
$\Delta 2^n = 1, 2, 4, 8, 16, 32, 64, 128, 256, ...$

It's the same! Why?

Consider the sequence n^2

$n^2 = 1, 4, 9, 16, 25, 36, 49, 64, 81, \ldots$

$\Delta n^2 = 3, 5, 7, 9, 11, 13, 15, 17, \ldots$ one can see that this is $2n + 1$

$\Delta(2n + 1),\ \Delta^2 n^2 = 2, 2, 2, 2, 2, 2, 2, 2, 2, \ldots$

$\Delta^3 n^2 = 0, 0, 0, 0, 0, 0, 0, 0, \ldots$ AS we can see this is very similar to differentiating

This sequence has a degree of 2. One can obtain the degree of the polynomial sequence as the following

$\Delta^{p+1} a = 0$, so hence the above is a degree of 2

One should observe, why do we get $2n + 1$ in the second difference?

Falling powers obtain us a factorial.

$n^k = n(n - 1)(n - 2)\ldots(n - k + 1),\ n^0 = 1$

Holds true for $k > 0$

Generalisation:

$\frac{n!}{(n-k)!}$

Fact: $\Delta n^k = kn^{k-1}$ like the power rule

Applying the generalisation to n^2

$$n^{\underline{1}} = n^1 = n$$

$$n^{\underline{2}} = n^2 - n^{\underline{1}}$$

From the generalisation

$$n^{\underline{k}} = \frac{n!}{(n-k)!}$$

So

$$n^2 = n^{\underline{2}} + n^{\underline{1}}$$

The powers fell, kind of like differentiating

Let us talk about the definition of e.

We know that $e^{x} = \sum_{k=0}^{\infty} \frac{x^{k}}{k!}$

So $e = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + ...$

But this is the continuous definition. What would the discrete definition be?

Discrete form being

$$\sum_{k=0}^{\infty} \frac{n^{\underline{k}}}{k!}$$

Recall that $n^{\underline{k}} = \frac{n!}{(n-k)!}$

If we substitute for the falling power definition we obtain

$$\sum_{k=0}^{\infty} \frac{n!}{k!(n-k)!}$$

which is the same as

$$\sum_{k=0}^{\infty} \binom{n}{k}$$

Evaluating at $n = 1$

$$1 + n + \frac{1}{2}n(n-1) + \frac{1}{6}n(n-1)(n-2) + ...$$

notice that everything with $n-1$ and above cancels as we substitute in 1. So we obtain

$$1 + 1 = 2$$

however recall this was at $n = 1$, so the generalization is

$$2^{n}$$

So 2^{n} is like the euler's number of discrete calculus.

Symbol	Name	Function	Examples
I	Identity	$(Ia)_n = a_n$	$I^2 = I$
E	Forward shift	$(Ea)_n = a_{n+1}$	$E^2, E^3, E^4 ... etc$
E^{-1}	Backward shift	$(E^{-1}a)_n = a_{n-1}$	$E^{-2}, E^{-3}, E^{-4} ... etc$
Δ	Forward difference	$\Delta a_n = a_{n+1} - a_n$	$\Delta = E - I$
∇	Backward difference	$\nabla a_n = a_n - a_{n-1}$	$\nabla = I - E^{-1}$

For higher derivatives, in which for example

$(\Delta^2 a)_n = \Delta(\Delta a)_n$

$= \Delta(a_{n+1} - a_n) = \Delta a_{n+1} - \Delta a_n$

$= a_{n+2} - a_{n+1} - (a_{n+1} - a_n)$

$= a_{n+1} - 2a_{n+1} + a_n$

OR, for example,
$\Delta^2 = (E - I)^2$
$= E^2 - 2EI + I^2$
Recall that the identity function does nothing when applied
$= E^2 - 2E + I$
$= a_{n+2} - 2a_{n+1} + a_n$

So one could come with a generalisation of
$\Delta^k = (E - I)^k$
Which would give a binomial relation of
$\sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} E^i$

Suppose a question asks to find the coefficient of the 6th term in the expansion of 8. Given that $i = 0$, that means the 6th will be $i = 5$.
So
$(-1)^{8-5} \binom{8}{5} E^5 = -56$

Lastly, consider the following
$\Delta = E - I$
$\Delta^{-1} = (E - I)^{-1}$
One would make it convenient to write it in the following form
$\Delta^{-1} = -(I - E)^{-1}$
Which is
$\Delta^{-1} = -\frac{1}{I-E}$. Reminding of the infinite geometric series formula. The expansion of such would be
$= -(I + E + E^2 + E^3 + E^4 + E^5 + \ldots.)$
Let us consider an example
$a = 3, -1, 4, -1, 5, -9, 2, -6, 5, 0, 0, 0, 0, \ldots$
$\Delta^{-1}a = -2, 1, 0, 4, 3, 8, -1, 1, -5, 0, 0, 0, 0, \ldots$

Otherwise:
$0 + 5 = 5 \rightarrow 5 \times -1 = -5.$
$0 + 5 - 6 = -1 \rightarrow -1 \times -1 = 1$
$0 + 5 - 6 + 2 = 1 \rightarrow 1 \times -1 = -1$ from the right side and so on.

Otherwise, we can define in terms of Δ
$\Delta = a_{n+1} - a_n$
$-(a_n + a_{n+1} + a_{n+2} + ...)$

However, notice that this only works for terms with discontinuity i.e. the ones that follow with 0s after.

Let us consider $\Delta(\Delta^{-1}a)$
You will notice that you will obtain
$\Delta(\Delta^{-1}a) = 3, -1, 4, -1, 5, -9, 2, -6, 5, 0, 0, 0, 0, ...$

Discrete calculus

Discrete function ⇒ Sequence
Discrete derivative ⇒ Difference
Discrete Integral ⇒ Series
Discrete differential equation ⇒ Recursion relation

The fundamental theorem of integral calculus

$$\sum_{n=a}^{b} \Delta u_n = [u_n]_a^{b+1}$$

Why?

Let us compute the Δu_n

$$\sum_{n=a}^{b} \Delta u_n = (u_{n+1} - u_n) + (u_{n+2} - u_{n+1}) + ... + (u_b - u_{b-1}) + (u_{b+1} - u_b)$$

Notice that everything cancels out except $-u_n$ and u_{b+1}.

For example, consider

$$\sum_{n=1}^{2} \Delta u_n = \Delta u_1 + \Delta u_2 = (u_2 - u_1) + (u_3 - u_2) = u_3 - u_1, \text{ true}$$

Example, consider $n!$
$\Delta n! = (n+1)! - n! = n!(n+1-1) = n!n$

$$\sum_{n=a}^{b} n!n = [n!]_a^{b+1}$$

Integration by parts

$$\sum_{n=a}^{b} uv = [uv]^{b+1}{}_{a} - \sum_{n=a}^{b} Ev\Delta u$$

Example

$$\sum_{n=0}^{k} n2^{n}$$

Let $u = n$ & $v = 2^{n}$, then

$= \left[n2^{n}\right]^{b+1}{}_{n} - \sum_{n=a}^{k} 2^{n+1}\Delta n$

$\Delta n = 1$ and $E2^{n} = 2^{n+1}$so

$= \left[n2^{n}\right]^{k+1}{}_{n} - \left[2^{n+1}\right]^{k+1}{}_{a}$

$= (k - 1)2^{k+1} + 2$

Differential equations

Consider

$u_{n+1} = \lambda u_{n}$

Let us rewrite

$u_{n+1} = Eu_{n}$

$Eu_{n} = \lambda u_{n}$

$(E - \lambda I)u_{n} = 0$

So, $u = C\lambda^{n}$ where C is the initial condition.

Which makes sense. The higher term is defined as lambda times the lower sequence, making a difference of λ. and this can be generalised as λ^{n}. One should not forget the initial constant, though.

Consider

$\Delta u = \lambda u$

Let us rewrite

$(E - I)u = \lambda u$

$(E - (\lambda + 1)I)u = 0$

Hence one obtains the solution as

$u = C(\lambda + 1)^n$ as that's the difference between E & I as one would obtain a zero.

Infinite series

Let an infinite series take the form

$$\sum_{n=1}^{\infty} a_n$$

Then one, using the same methodology of improper integrals, could rewrite

$$\lim_{T \to \infty} \sum_{n=1}^{T} a_n$$

However, one needs to determine convergence or divergence of an infinite series.
Consider

$$\sum_{n=0}^{\infty} e^{-n} = 1 + \frac{1}{e} + \frac{1}{e^2} + \frac{1}{e^3} + \cdots = \frac{1}{1-e^{-1}}$$

One can notice that this is a geometric series. However,
Consider the function

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

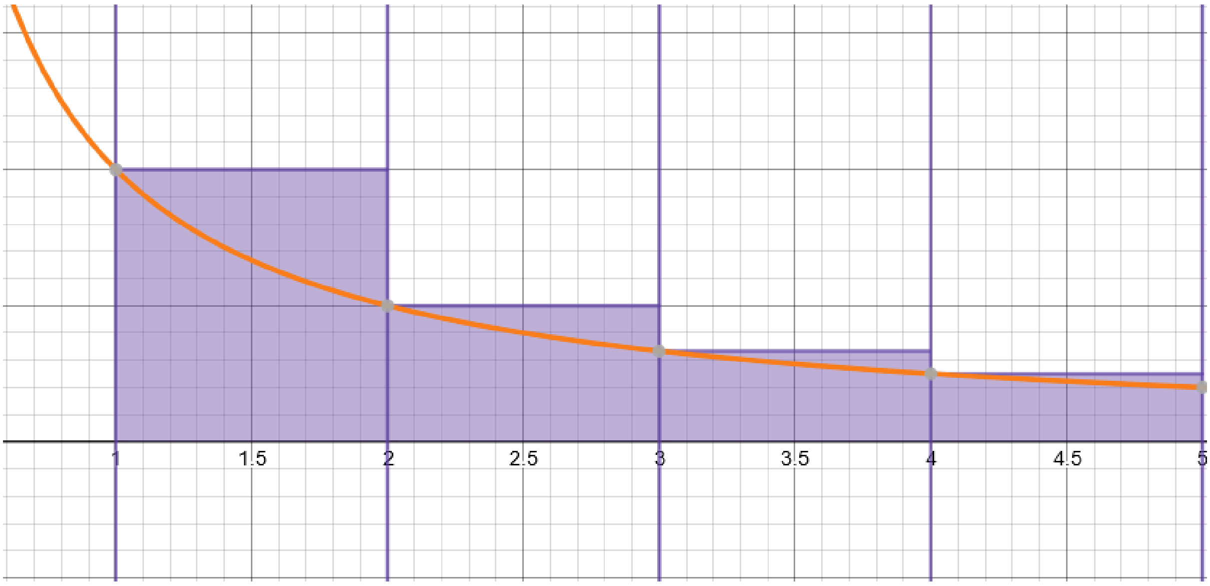
One would ask, does it diverge or converge?
Consider the following

$$\sum_{n=1}^{\infty} \frac{1}{n} \sim \int_{1}^{\infty} \frac{1}{n}$$

Recall that (from the improper integral section)

$\int_{1}^{\infty} x^{-p}$ for $p \leq 1 \Rightarrow$divergence

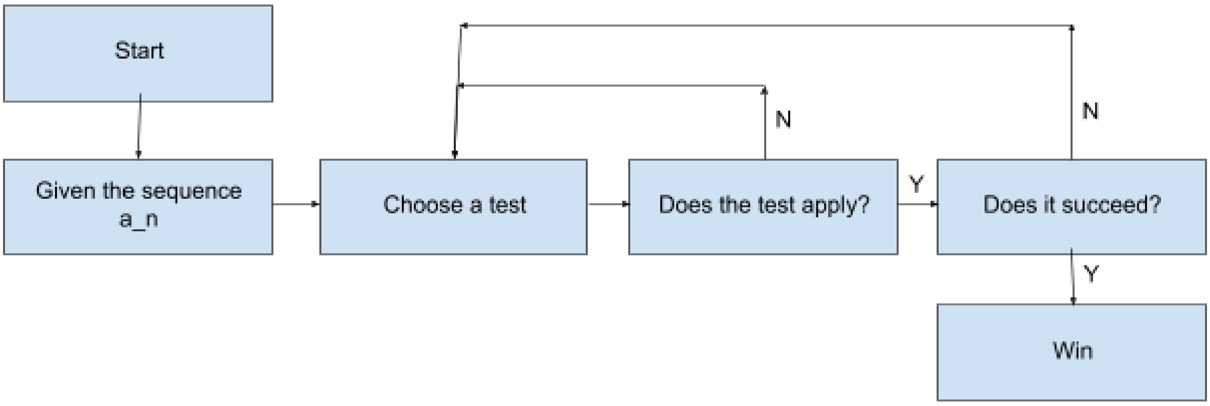
Given that n^{-p} & $p = 1$, then divergence occurs.
One could think of the infinite series as a 'discretisation' of such an integral.
For a more clear image, consider the graph of $\frac{1}{x}$



What the sum inherently does is that it creates a Riemann sum with step size 1 (i.e. width 1) with height determined by the intersection of the left side. By this interpretation, then one can effectively conclude that

$$\sum_{n=1}^{\infty} \frac{1}{n} > \int_{1}^{\infty} \frac{1}{n}$$

There are also tests to determine whether something convergences or diverges. Consider the following diagram



The first test to learn is called the nth term test

If $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{n}^{\infty} a_n$ diverges

E.g.

$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ then $\lim_{n\to\infty} \frac{(-1)^n}{n!} = 0$, hence it converges

Of course, we know this is true because the sum equals e^{-1} from definition of taylor and sums

However, consider the harmonic series
$\sum_{n=0}^{\infty} \frac{1}{n}$, then $\lim_{n\to\infty} \frac{1}{n} = 0$. Does this mean it converges? No.

One needs to consider the logical statement
If X then Y
I.e.
If $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{n}^{\infty} a_n$ diverges
One CANNOT do
If Y then X
I.e.
If $\sum_{n}^{\infty} a_n$ diverges then $\lim_{n\to\infty} a_n \neq 0$

I.e. one cannot conclude that if the sequence diverges that the limit will be always be not 0. However, one can conclude that if the limit is not 0 then it is divergent.
This is called the converse, and this will not always be true.

What holds, however , is the contrapositive of the statement.
If NOT Y then NOT X
I.e.
If $\sum_{n}^{\infty} a_n$ converges then $\lim_{n\to\infty} a_n = 0$

<u>Convergence Tests I</u>

Integral test

One could consider expressing an infinite arithmetic series as an integral. I.e. one could think of an infinite arithmetic series as a discretized form of an improper integral.

A prerequisite for this test to hold true is that
$0 \leq a_{n+1} \leq a_n$
$a'(x) \leq 0 \leq a(x)$

$a(n) = a_n$

I.e. that the arithmetic sequence always decreases in higher terms. The derivative of the higher function is always decreasing, and such that all points of the continuous function touch the arithmetic series' individual values.

As such, the definition is as follows:

If $0 \leq a_{n+1} \leq a_n$
$a'(x) \leq 0 \leq a(x)$
$a(n) = a_n$

Then $\sum_{n}^{\infty} a_n \sim \int_{n}^{\infty} a(x)dx$
With convergence and divergence following an IFF statement (if and only if ⇔). This means the following:
If A then B
And If B then A
Moreover, the contrapositive would be
If NOT A then NOT B
If NOT B then NOT A
Otherwise $A \Leftrightarrow B$
For example:

Consider
$$\sum_{2}^{\infty} \frac{1}{n\ln^{p} n}$$
Writing it as an integral
$\int_{2}^{\infty} \frac{1}{n\ln^{p} n} = \frac{\ln^{1-p} n}{1-p}$, evaluating at ∞ and 2 gives
$< \infty, p > 1$
$\infty, p \leq 1$
Recall this from improper integral definition for convergence.

Comparison test

Two sequences a_n and b_n can be compared

If $0 \le a_n \le b_n$ then $\sum_n^\infty a_n \le \sum_n^\infty b_n$

Convergence $\Leftarrow$convergence

Divergence $\Rightarrow$ divergence

One could think of scaling a_n to a higher or a lower state. Since, for example, you are scaling it to a higher sequence, and that sequence converges, that means that the smaller must as well. However, if the bigger sequence diverges, it is possible that it has been upscaled so much that only the bigger sequence diverges hence it may not hold true for the smaller.

Example

$\sum_{n=0}^{\infty} \frac{\cos^2 n}{1+\sqrt{n^3}}$

Let $a_n = \frac{\cos^2 n}{1+\sqrt{n^3}}$. Let $b_n > a_n$.

This means the following

$\frac{\cos^2 n}{1+\sqrt{n^3}} \begin{smallmatrix} \le \\ > \end{smallmatrix} \frac{1}{\sqrt{n^3}}$

So consider

$\sum_{n=0}^{\infty} \frac{1}{\sqrt{n^3}} = \frac{1}{n^{\frac{3}{2}}}\ \frac{3}{2} > 1$, hence it converges. Thus the smaller must converge.

Limit test

If $0 < a_n$, $0 < b_n$ and $0 < \lim_{n\to\infty} < \infty$ then $\sum_n^\infty a_n \sum_n^\infty b_n$

Convergence $\Leftrightarrow$convergence

Divergence $\Leftrightarrow$ divergence

Basically, the leading order term must be equivalent.

Example

$\sum_n^\infty \ln^2(1 + \frac{1}{n^2}) \sim \sum_n^\infty (\frac{1}{n^2})^2 \sim \sum_n^\infty \frac{1}{n^4}$

$1 < 4$, so it converges.

Convergence Tests II

Root test

We will not consider convergence tests which effectively compare the series to a geometric series.
Consider the following geometric series
$$\sum_{n=0}^{\infty} x^{k} = 1 + x + x^{2} + x^{3} + \cdots$$
Rewriting as a limit
$$\lim_{T\to\infty} \sum_{n=0}^{T} = 1 + x + x^{2} + x^{3} + \cdots + x^{T}$$

This is the basis of the test.
$$\lim_{n\to\infty} \sqrt[n]{a_n}$$

If $0 < a_n$, $\rho = \lim_{n\to\infty} \sqrt[n]{a_n}$ then $\sum_{n}^{\infty} a_n$, converges for $p < 1$, diverges for $p > 1$ and is failing for $p = 1$

Consider
$$\sum_{n=1}^{\infty} (\ln(1 + \cos\frac{1}{n}))^{\frac{1}{2}n}$$
$$\rho = ((\ln(1 + \cos\frac{1}{n}))^{\frac{1}{2}n})^{\frac{1}{n}}$$
$$\rho = \sqrt{\ln(1 + \cos\frac{1}{n})}$$
$$\rho = \lim_{n\to\infty} (\sqrt{\ln(1 + \cos\frac{1}{n})})$$
$\rho = \sqrt{\ln 2} < 1$, hence convergent.
It is also important to remember that
$$\lim_{n\to\infty} \sqrt[n]{n} = 1$$

Ratio test

Ratio tests check the difference between the original and leading term in a manner of ratios.

If $0 < a_n$, $\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ otherwise $\rho = \lim_{n\to\infty} (a_{n+1} \times \frac{1}{a_n})$then $\sum_n^\infty a_n$, converges for $p < 1$, diverges for $p > 1$, and fails for $p = 1$

Consider

$$\sum_{n=0}^{\infty} \frac{n^n}{n!}$$

Let $a_n = \frac{n^n}{n!}$, then $a_{n+1} = \frac{(n+1)^{n+1}}{(n+1)!}$

$$\lim_{n\to\infty} \frac{(n+1)^{n+1}}{(n+1)!} \times \frac{n!}{n^n}$$

Consider cancelling the factorials to obtain

$$\frac{(n+1)^{n+1}}{(n+1)} \times \frac{1}{n^n}$$

More cancellation can occur;

$$\lim_{n\to\infty} \frac{(n+1)^n}{n^n}$$

Notice the following

$$\lim_{n\to\infty} (\frac{n+1}{n})^n$$

$$\lim_{n\to\infty} (1 + \frac{1}{n})^n = e$$

$e > 1$, hence divergent

- [ ] $$\sum_{n=1}^{\infty} \left(\frac{3n}{3n-1}\right)^{n^2}$$
- [x] $$\sum_{n=1}^{\infty} \left(\frac{3n-1}{3n}\right)^{n^2}$$

First question

Applying root test to obtain

$$\lim_{n\to\infty} (\frac{3n}{3n-1})^n$$

Attempting to obtain e form

$$\frac{3n-1+1}{3n-1} = 1 + \frac{1}{3n-1}$$

$\lim_{n\to\infty} (1 + \frac{1}{3n-1})^n$

Recall from definition of e

$\lim_{n\to\infty} (1 + \frac{x}{n})^n = e^x$

So one could say that $x = \frac{1}{3}$, whilst the -1 is a part of $O(n)$

Hence, $= e^{\frac{1}{3}}$, so divergent

Question 2

$\lim_{n\to\infty} (1 - \frac{1}{3n})^n$

So $x = -\frac{1}{3}$

Hence $= e^{-\frac{1}{3}}$

Thus convergent

Absolute and Conditional convergence

The following only applies for for alternating series, generally of the form

$\sum_{n}^{\infty}(-1)^n a_n$

If $0 \leq a_n \leq a_{n+1}$ then $\lim_{n\to\infty} a_n = 0 \Leftrightarrow$convergence

$\lim_{n\to\infty} a_n \neq 0 \Leftrightarrow$ divergence

There are 2 important definitions to follow

Absolute convergence implies that both $\sum_{n}^{\infty} a_n$ AND $\sum_{n}^{\infty}|a_n|$ converge

Conditional convergence implies that $\sum_{n}^{\infty} a_n$ converges BUT $\sum_{n}^{\infty}|a_n|$ diverges

Recall

$\ln(1 + x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^k}{k}$ for $|x| < 1$. However, is that REALLY true?

Consider $x = 1$

$\sum_{k=0}^{\infty}(-1)^{k}\frac{1}{k}$ is the harmonic alternating series, and evaluating limits from our definition gives 0. Hence this is convergent. However, the absolute value of this would give the regular harmonic series, and that is divergent.

Power series

Power series is a function expressed in the form

$\sum_{n=0}^{\infty} a_n x^n$

So a_n is now the coefficient.

Consider the fibonacci sequence

Let $F(x) = \sum_{n=0}^{\infty} C_n x^n$

So

$F(x) = C_0 + C_1x + C_2x^2 + C_3x^3 + C_4x^4 + \cdots + C_nx^n + \cdots$

$xF(x) = \quad C_0x + C_1x^2 + C_2x^3 + C_3x^4 + \cdots + C_{n-1}x^n + \cdots$

$x^2F(x) = \quad\quad C_0x^2 + C_1x^3 + C_2x^4 + \cdots + C_{n-2}x^n + \cdots$

Hence one obtains the recursion relation

$F_n = F_{n-1} + F_{n-2}$

When subtracting one obtains

$C_0 + (C_1 - C_0)x + 0x^2 + 0x^3 + 0x^4 + \cdots + 0x^n + \cdots$

Because of the recursion relation

We, however know that $C_0 = 0$ & $C_1 = 1$

So

$(1 - x - x^2)F(x) = x$

So

$F(x) = \frac{x}{1-x-x^2}$

Such functions share many properties, such as sums, integrals, convergences, etc.

Convergence

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$, for some $0 \leq R \leq \infty$

$f(x)$ converges absolutely if $|x| < R$

And diverges if $|x| > R$

One considers the ratio test when trying to find the ratio of convergence of a power series.

$$\rho = \lim_{n\to\infty} \frac{|a_{n+1}x|}{|a_n|} = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} x < R$$

<1 abs. Convergence

>1 divergence

One can see that if you were to take the reciprocal and move the coefficient a_n obtained to the RHS that one can say that

$$R = \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|},$$

$\rho \Leftrightarrow |x| < R$ convergent

$\rho \Leftrightarrow |x| > R$ divergent

Shifted power series

They take the form

$$f(x) = \sum_{n=0}^{\infty} a_n (x - c)^n$$

For example

$$f(x) = \sum_{n=1}^{\infty} \frac{(3x-2)^n}{n4^n}$$

In an attempt to achieve the form above, we do the following

$$\frac{(3(x-\frac{2}{3}))^n}{n4^n}$$

$$= \frac{3^n(x-\frac{2}{3})^n}{n4^n}$$

Thus

$a_n = \frac{1}{n}(\frac{3}{4})^n$

$R = \lim_{n\to\infty} \frac{n+1}{n}\frac{4}{3} = \frac{4}{3}$ at center $x = \frac{2}{3}$

Taylor Series Redux

One can consider Taylor series as a way of turning a function into a power series.
Theorem:

If $f(x) = \sum_{n=0}^{\infty} a_n x^n = (\sum_{n=0}^{\infty} \frac{1}{n!} f^n(0) x^n$

Then

f converges absolutely on $|x| < R$, recall $R = \frac{|a_n|}{|a_{n+1}|}$

For Taylor, that is $R = (n + 1)\frac{f^n(0)}{f^{n+1}(0)}$

Within the domain of convergence:

f is differentiable: $\frac{df}{dx} = \sum_{n=0}^{\infty} n a_n x^{n-1}$

f is integrable: $\int f(x)dx = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$

These can be used to compute hard integrals such as the error function, fresnel integrals, hypergeometric functions etc.

Approximation and Lagrange Error

Approximation is used in order to approximate the final value of an infinite sum.

Consider the following

$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{N} a_n + E_N$

Where E_N is the error term

Now, consider

$\sum_{n}^{\infty}(-1)^n a_n = a_0 - a_1 + a_2 - a_3 + a_4 - a_5 + \cdots = \lim_{n\to T} \sum_{n=0}^{T} (-1)^n a_n$

So brings up the theorem:

If $0 \leq a_{n+1} \leq a_n$, $\lim_{n\to\infty} a_n = 0$, then $|E_n| \leq a_{n+1}$

Because you are overshooting each time, one could say that the error term of the current position of sequence is closer to the actual answer than the next term, as it will go past it.

Example

Consider the following

$\frac{1}{\sqrt{e}}$ and you want to approximate to 0.001.

$$\frac{1}{\sqrt{e}} = e^{-\frac{1}{2}} = \sum_{n=0}^{\infty} \frac{(\frac{1}{2})^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!2^n}$$

$$|E_N| \leq a_{n+1} < 0.001$$

$$\frac{1}{(N+1)!2^{N+1}} < 0.001$$

True when $(N+1)!\,2^{N+1} > 1000$, so $N \geq 4$

Integral approximation

Theorem:

If $0 \leq a_{n+1} \leq a_n$, $a'(x) \leq 0 \leq a(x)$ and $a(n) = a_n$

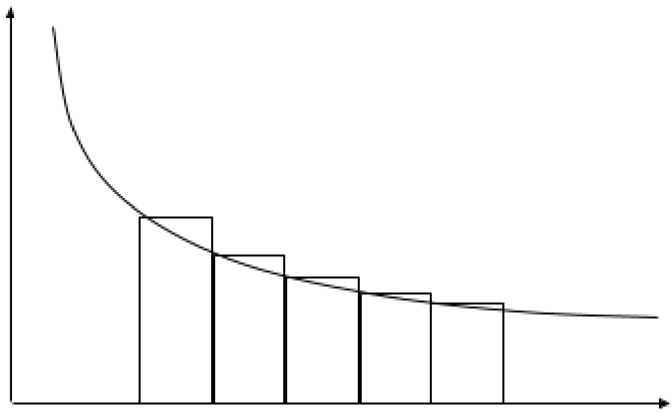
Then

$$\int_{N+1}^{\infty} a(x)dx < E_N < \int_{N}^{\infty} a(x)dx$$

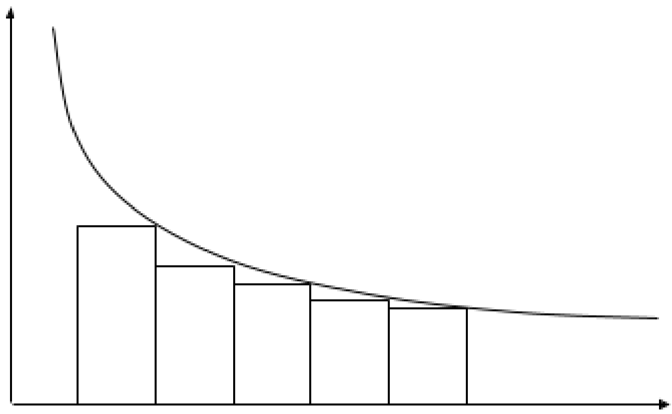
Otherwise

$$\sum_{N+1}^{\infty} a_n < E_N < \sum_{N}^{\infty} a_n$$

The left integral is the a Riemann sum with left upper boundaries, consider the following (the left integral):



The right integral on the other hand is, basically everything shifted 1 to the left as per translation of the limit (the right integral):



E.g.
You are given

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} = 1.6449...$$

And you wanna compute $E_N < 0.001$

$$a(x) = \frac{1}{x^2}$$

$$E_N < \int_N^{\infty} \frac{1}{x^2} < 0.001$$

$$E_N < \frac{1}{N} < 0.001$$

So $N > 1000$

Taylor series and Lagrange error

Theorem:
If f is smooth (that is, a continuously differentiable function forever like $\sin x$) for x close to 0, then

$$f(x) = \sum_{n=0}^{N} \frac{1}{n!} f^n(0)x^n + E_N(x)$$

Weak error term: $E_N(x)$ is $O(x^{N+1})$

Lagrange error term:
$\left|E_N(x)\right| < \frac{f^{N+1}(t)}{(N+1)!}x^{N+1}$ for some t between 0 and x

Example

$\sqrt{e}, = \sum_{n=0}^{\infty} \frac{1}{n!}(\frac{1}{2})^n$
$= \frac{1}{N!}\frac{1}{2^N} + E_N$
Where $E_N < \frac{C}{(N+1)!2^{N+1}}$

$\frac{d}{dx}^{N+1} e^x < C,\ 0 \leq x \leq \frac{1}{2}$
$e^x < C < e^{\frac{1}{2}} < 2$
So
$E_N < \frac{C}{(N+1)!2^{N+1}} < \frac{1}{(N+1)!2^N}$
I.e. $C = 2$ because $e^{\frac{1}{2}}$ is close to 2. A good upper bound is $x = \frac{1}{2}$ because e is exponential and thus taking the highest would be the most appropriate

<u>Logical Statements</u>

IF, THEN (⇒)

IFF, IF AND ONLY IF (⇔)

CONTRAPOSITIVE

CONVERSE

<u>The proof of</u> $\lim_{n\to\infty} (1 + \frac{x}{n})^n = e^x$

Proof 1
Consider the following
$\lim_{n\to\infty} e^{n\ln(1+\frac{x}{n})}$

The Maclaurin for $\ln(1 + x)$ can be recalled as the following

$$\sum_{k=0}^{\infty}(-1)^k \frac{x^{k+1}}{k+1} = x + O(x^2)$$

Now we know that the first term will be then the following

$$\lim_{n\to\infty} e^{n(\frac{x}{n}+O(\frac{x^2}{n^2}))}$$

$$\lim_{n\to\infty} e^{x+O(\frac{x^2}{n})}$$

$$= e^x$$

Proof 2

Consider binomially expanding $(1 + \frac{x}{n})^n$

$$(1 + \frac{x}{n})^n = 1 + \frac{n}{1!}\frac{x}{n} + \frac{n(n-1)}{2!}\frac{x^2}{n^2} + \frac{n(n-1)(n-2)}{3!}\frac{x^3}{n^3} + \cdots$$

$$= 1 + x + \frac{n^2-n}{n^2}\frac{x^2}{2!} + \frac{n^3-3n^2+2n}{n^3}\frac{x^3}{3!} + \cdots$$

Evaluating the limit

$$\lim_{n\to\infty}(1 + x + \frac{1-\frac{1}{n}}{1}\frac{x^2}{2!} + \frac{1-\frac{3}{n}+\frac{2}{n^2}}{1}\frac{x^3}{3!} + \cdots)$$

$$= (1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots) = \sum_{k=0}^{\infty}\frac{x^k}{k!} = e^x$$

Proof 3

$$\lim_{n\to\infty} y = \lim_{n\to\infty}(1 + \frac{x}{n})^n$$

$$\ln y = n\ln(1 + \frac{x}{n})$$

$$\ln y = \frac{\ln(1+\frac{x}{n})}{\frac{1}{n}}$$

Applying l'Hopital

$$\frac{d}{dn}\frac{\ln(1+\frac{x}{n})}{\frac{1}{n}} = \frac{-\frac{x}{n^2}\frac{1}{1+\frac{x}{n}}}{-\frac{1}{n^2}} = x \cdot \frac{1}{1+\frac{x}{n}}$$

$$\lim_{n\to\infty}\ln y = \lim_{n\to\infty} x \cdot \frac{1}{1+\frac{x}{n}}$$

$\ln y = x$

$y = e^{x}$

Epsilon-Delta definition of a derivative

The fundamental theorem of integral calculus and Riemann sums

A Riemann integral which is defined is given as the following

$\sum_{i=1}^{n} f(x_i)(\Delta x)_i$, whilst the theorem states that $\lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i)(\Delta x)_i$

Where Δx is the 'width' of the sum. A uniformly defined width when computing sum from the intervals $[a, b]$ would be $\frac{b-a}{n}$, also called the 'partition'.

x_i is the sampling, or rather the height. A uniform height can be chosen by $a + i\frac{b-a}{n}$. I.e. you take the initial value a and then add the interval of partition to it, depending on nth square (i.e. assume you split the value into 5 pieces, that is $n = 5$. You want to calculate the height of the third rectangle, so the corresponding x value will be $a + 3\frac{b-a}{5}$). E.g. if its split into 2 you get an interval from $i = 1$ and $i = 2$. Because there are 2 squares. At $n = 2$ and $i = 2$ one will notice that they both will cancel, and thus it can be deducted one will compute only b, (that is ,$a + 2\frac{b-a}{2}$, hence it will be $y = f(b)$ and this will hold true when $i = n$, which is always the last 'square' or 'rectangle' no matter the chosen n, showing that we are choosing and using the formula for the right endpoint Riemann sum).

Then once getting the sampling or the 'height' you can substitute it into the function, which will give you the corresponding y values of the height for each rectangle respectively. One has to repeat this for every rectangle, and then add the rectangles up, hence the sum formula for each i (for each rectangle, basically).

https://mathinsight.org/calculating_area_under_curve_riemann_sums

https://www.youtube.com/watch?v=FZKRsD9FqU4

ELSE

The complete elliptic integral of the first kind is defined as

$$\int_{\theta=0}^{\frac{\pi}{2}} \frac{\mathrm{d}\theta}{\sqrt{1-k^2\sin^2\theta}}$$

$$\int_{t=0}^{1} \frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-k^2t^2)}}$$

And is an example of a hypergeometric function. These can be used to solve second-order linear differential equations