IB Mathematics

Internal Assessment

Calculus of Variations

- Would the function for the minimal surface area of a revolution be a good function to model a soap film? -

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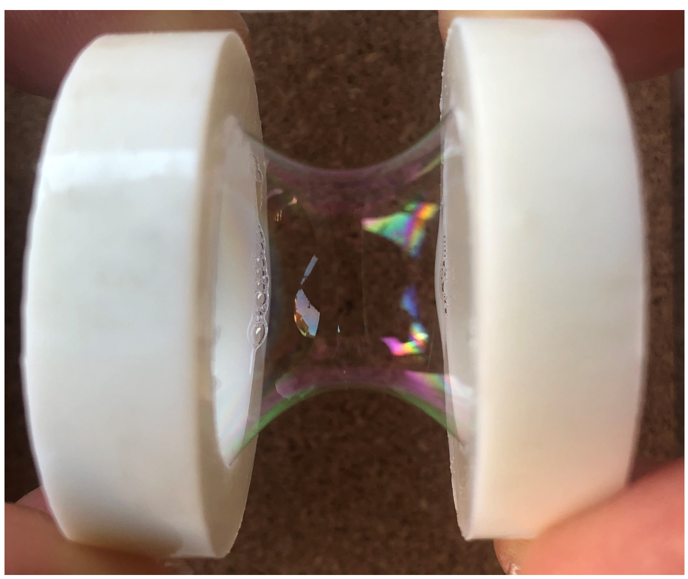

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Candidate #

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1 Introduction

As a person who has been always intrigued with the mathematical nature of objects within our life, my reason to pursue further education within college is to explore and define the relationship between mathematics, computer science and the world. One of the key components of computer science for me is its ability to mimic real life, given that it writes its rules using the language of mathematics. To explore this idea, in this internal assessment I will be focusing on exploring and mathematically defining the behavior of some material, as I hope to integrate my findings to strengthen my understanding on how the real world can be virtually simulated. Therefore in this Internal Assessment, I will be exploring a material which is one of its kind: soap films. The idea of exploring soap films came to me from my Single Variable Calculus MOOC (Ghrist, 2016) that I took over the summer out of interest. The professor mentioned the intriguing property of the $\cosh(x)$ function as we were exploring hyperbolic $u-$substitutions for integration, briefly mentioning that it hides an interesting property in real life. After short research, I have found out that the intriguing function $\cosh(x)$ is involved in various minimal shapes and surfaces. To see this in action, I will be looking at soap films, an ideal candidate to test out any minimal surfaces for my exploration as it is a natural material that always is a minimal of surfaces (*Introduction to the calculus of variations*, 2016, p.17-18). I have decided to infuse the idea of soap films and minimal surfaces with a recent topic we have covered in class of rotating functions around an axis to obtain interesting and complex shapes. Therefore in this Internal Assessment I seek to find whether the soap film would replicate my results after I find the minimal surface area of a revolution in the hope of answering the question ”Would the function for the minimal surface area of a revolution be a good function to model a soap film?”. However, finding the minimum of the surface area of a revolution is no easy task, and I will require a lot of theory before being able to answer my exploration. Namely, I will first have to find the equation that will represent the surface area of a revolution of a function in some domain and effectively find the minimal of this equation using the Beltrami Identity, a special case of the Euler-Lagrange equation from Calculus of Variations which I require to understand first.

2 The theory of the investigation

2.1 Arc length Integral

I first require to understand what equation I will be truly minimising. Intuitively the equation must include the arc length of the function as I will rotate it along an axis, similar to volume of revolution

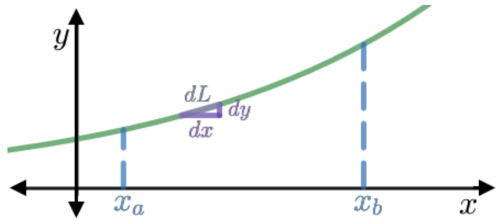

Figure 1: A graphical depiction of infinitesimal triangle $dydx$ (not to scale)(*Desmos Graphing Calculator*, 2011)

To compute the surface area of a revolution of some function $f(x)$ in the domain (x_a, x_b) where $x_a < x_b$ and $x_a, x_b \in \mathbb{R}$, I will consider the triangle of an infinitesimally small width dx and its corresponding infinitesimally small height dy, to then compute its infinitesimally small hypotenuse dL which represents an infinitesimally small function length by Pythagoras' Theorem.

$$dL = \sqrt{dy^2 + dx^2}$$

And if I factor out dx from the square root I can obtain

$$dL = \sqrt{\left(\frac{dy}{dx}\right)^2 + 1}\, dx \tag{1}$$

Then, if I use the definite integral definition to be the sum of all infinitesimals within the domain (x_a, x_b), I can obtain the exact length of the function as

$$\boxed{L = \int_{x_a}^{x_b} \sqrt{\left(\frac{dy}{dx}\right)^2 + 1}\, dx}$$

2.2 Surface Area Of A Revolution

Whilst I have found a way to compute the arc length of a function, I will now have to find a way to rotate it in order to obtain the surface area of a revolution. If I consider the volume of revolution integral V in the domain (x_a, x_b)

$$V = \pi \int_{x_a}^{x_b} f(x)^2 dx$$

Which is indeed derived from the classical formula for the area of a circle πr^2, where the radius r can be represented by an infinitesimal rectangle of width dx (imitating a very thin strip) and height $f(x)$. If the infinitesimal rectangle is rotated 360° around the x axis, it forms an infinitesimal cylinder of width dx (which in turn imitates a circle, hence I use the formula for an area of a circle). The integral sums all such infinitesimal cylinder areas in the domain of its bounds to obtain the volume.

Using similar intuition, I can use the formula $2\pi R$ to sum the surface areas of the infinitesimal cylinders instead. However, in this particular case, the R within the function is not represented by $f(x)$ only. If I use the same logic to rotate a rectangle of width dx and height $f(x)$ around the x axis to obtain the same cylinder, I obtain no information on the surfaces. Instead, I need something that can represent this surface as I rotate. If I introduce the infinitesimal arc length from equation 1 to be included after the height $f(x)$ at some point, I am obtaining a new shape by the name of a "frustum" (a shape similar to a cone, but the cone's tip is subtracted by a smaller similar cone). A frustum's surface area can be conveniently defined through its base radius multiplied by its arc length and 2π (*Areas of Surface of Revolution*, 2019). Infinitesimally, I then can define R

$$R = f(x)\sqrt{\left(\frac{dy}{dx}\right)^2 + 1}\, dx$$

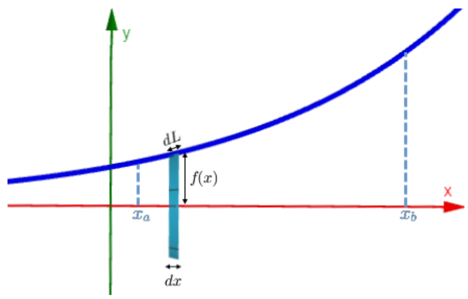

Figure 2: An infinitesimal frustum with width dx (not to scale)(*Geogebra 3D Calculator*, 2011)

If I substitute my R into the circumference formula and sum all infinitesimal surface areas (like summing infinitesimal circumferences of circles) in the interval (x_a, x_b) by a definite integral, the surface area of a revolution is then

$$\boxed{S = 2\pi \int_{x_a}^{x_b} f(x)\sqrt{\left(\frac{dy}{dx}\right)^2 + 1}\, dx} \tag{2}$$

2.3 Hyperbolic Trigonometric Functions

Since the professor has mentioned that this property is specifically held within the hyperbolic trigonometric functions as I was doing various u substitutions, it is wise define these functions and their identities, as I am likely to require them in the future. The hyperbolic trigonometric functions are defined to be (Weisstein, n.d.)

$$\sinh(x) \equiv \frac{e^x - e^{-x}}{2}$$
$$\cosh(x) \equiv \frac{e^x + e^{-x}}{2}$$

The inverse of these functions are defined to be $\operatorname{arsinh}(x)$ and $\operatorname{arcosh}(x)$ respectively. Furthermore, the hyperbolic functions result in the identity

$$\boxed{\cosh^2(x) - \sinh^2(x) \equiv 1} \tag{3}$$

Which is an identity similar to $\tan^2(x) - \sec^2(x) \equiv 1$, allowing an alternative for substitution for integrals originally requiring $\tan(x)$, and can simplify some work which is the case for me later in the investigation. I will show the validity of this identity by substituting the algebraic definitions of the functions, thus

$$\implies \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$$
$$\implies \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4} = \frac{4}{4} = 1$$

Lastly, knowing the algebraic definitions of $\cosh(x)$ and $\sinh(x)$, it is also implied that $\cosh(x)$ is even and $\sinh(x)$ is odd, that is:

$$\cosh(-x) = \frac{e^{-x} + e^x}{2} = \cosh(x)$$

and

$$\sinh(-x) = \frac{e^{-x} - e^x}{2}$$
$$\implies -\frac{e^x - e^{-x}}{2} = -\sinh(x)$$

2.4 Euler-Lagrange Equation

The Euler-Lagrange Equation is an equation to find extrema (maximas and minimas) of equations I of the kind

$$I = \int_{x_a}^{x_b} F(x, y, y') \tag{5}$$

The integrand contains what is defined to be a functional, which is a function with functions as its input. Indeed, the surface area of a revolution formula is also an equation with an integrand that is explicitly dependent on y and y', therefore the Euler-Lagrange is applicable for my investigation. The Euler-Lagrange equation is defined as

$$\boxed{\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0} \tag{6}$$

However, this equation does not state the nature of the extrema it finds. The notation ∂ used within the equation denotes a partial derivative, these allow me to analyse the rate of change of a multivariate function with respect to a single variable, treating the rest like constants. For example, if I consider the equation $F(x, y) = x^3y + y^3x$, its partial derivative with respect to y is

$$\frac{\partial F}{\partial y} = x^3 + 3y^2x$$

As I want to understand how the Euler-Lagrange equation works as it is a tool that is useful in this exploration, I will see how it will help me find the minimum of a surface area of a revolution by analysing its derivation. However, in order to do this, I will take the multivariate chain rule as a given. The multivariate chain rule states that for some multivariate function $F(x_1, x_2, \ldots, x_n)$ with distinct variables $x_1, x_2, \ldots, x_n$, when differentiated with respect to some variable such as t, the chain rule can be given by

$$\frac{dF}{dt} = \sum_{i=1}^{n} \frac{\partial F}{\partial x_i} \frac{dx_i}{dt}$$

Similarly, if I am looking for the partial derivative instead

$$\frac{\partial F}{\partial t} = \sum_{i=1}^{n} \frac{\partial F}{\partial x_i} \frac{\partial x_i}{\partial t}$$

The rigorous proofs of these generalisations, however, are well beyond the scope of this IA.

To understand how the Euler-Lagrange specifically applies to the equation of surface area of a revolution, I will begin by denoting the function (*The Euler-Lagrange Equation, or Euler's Equation*, n.d.)

$$\boxed{Y(x) = y(x) + \varepsilon\eta(x)} \tag{7}$$

Where $y(x)$ is a twicely differentiable function whose domain is $[x_a, x_b]$ with points (x_a, y_a) and (x_b, y_b). For $y(x)$ I will assume that it is an extremum function in the interval $[x_a, x_b]$ (that is, it is either a function that maximises or minimises of my function $Y(x)$ in the domain $[x_a, x_b]$).

$\eta(x)$ is a twicely differentiable function whose domain is $[x_a, x_b]$ and has coordinates $(x_a, 0)$ and $(x_b, 0)$. This allows me to add some arbitrary function to $y(x)$ (variation) in order to get $Y(x)$ in any arbitrary form whilst ensuring my boundary conditions in (x_a, y_a) and (x_b, y_b) stay constant;

ε is some parameter which which allows me to configure my variation of $\eta(x)$.

In order to intuitively understand equation 7, I have plotted Figure 3 to visualise it in terms of arc length:

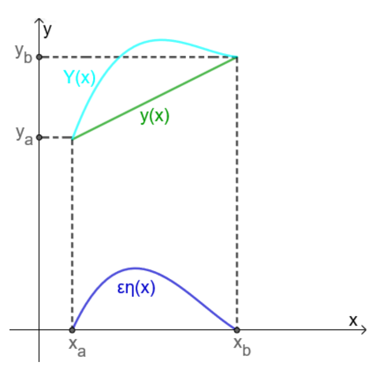

Figure 3: diagram of $Y(x), y(x)$ and $\varepsilon\eta(x)$ (*Geogebra 3D Calculator*, 2011)

My problem is then to find the extrema of the function $Y(x)$. Since $\eta(x)$ is some arbitrary function, it means that

$Y(x)$ will represent a family of curves which can be resulted from adding some arbitrary function $\eta(x)$ to an extremum $y(x)$.

Intuitively, a possible extremum function is a straight line in an arc length of the kind $y = mx + b$ (as seen in figure 3), this minimises distance between two points. Moreover, if I were to choose the parameter ε to be approaching smaller values (closer to 0), then function $Y(x)$ will approach the extremum $y(x)$ (that is, the variation added from the arbitrary function $\varepsilon\eta(x)$ will be getting smaller). If I consider the equation $Y(x)$ (equation 7) which represents twicely differentiable family of curves in a restricted domain (namely my domain where I seek to minimise, that is, the domain which I rotate around the x axis) and write it in the same as the equation of the kind 5, I obtain general integral of a specific family of curves. Namely:

$$I = \int_{x_a}^{x_b} F(x, Y, Y')dx$$

I know from the definitions of Y and Y' are dependent on x and ε. However, once the integral is computed with the x boundaries, only the variable ε is left in the equation. This means that I is actually only dependent on the parameter ε and its value, that is

$$I[\varepsilon] = \int_{x_a}^{x_b} F(x, Y, Y')dx$$

And to find a candidate for the minimum, as I is only dependent on ε, I can set its derivative to 0, in order to compute an extremum like in regular calculus. This allows me to find the extremum from the variational change added to $y(x)$

$$\frac{dI}{d\varepsilon} = 0$$

However, recall from my assumption that $y(x)$ is an extremum function, so I can then deduce that the solution of the equation above would be when I set the variation $\varepsilon = 0$ (when my family of functions becomes an extremum i.e. $Y(x) = y(x)$), giving me the equation

$$\left.\frac{dI}{d\varepsilon}\right|_{\varepsilon=0} = 0$$

substituting I

$$\frac{d}{d\varepsilon}\int_{x_a}^{x_b} F(x, Y, Y')\bigg|_{\varepsilon=0} dx = 0$$

The boundaries of the integral are constants. Furthermore, the integrand consists of a multivariate function whose variables can be reduced to x and ε only. The derivative, on the other hand, is with respect to ε. This implies that I can apply Leibniz integral rule given as (Haile, 2020)

$$\frac{d}{d\varepsilon}\left(\int_{x_a}^{x_b} f(x, \varepsilon)dx\right) = \int_{x_a}^{x_b} \frac{\partial}{\partial\varepsilon} f(x, \varepsilon)dx$$

Moreover, I know that x is not dependent on ε, so its derivative will simply be 0. However, $Y(x)$ and $Y'(x)$ are explicitly dependent on ε, which implies after I evaluate the RHS

For x, as x is not dependent on ε	For Y	For Y'
$$\frac{\partial F}{\partial x}\frac{\partial x}{\partial \varepsilon} = 0$$	$$\frac{\partial F}{\partial Y}\frac{\partial Y}{\partial \varepsilon}$$	$$\frac{\partial F}{\partial Y'}\frac{\partial Y'}{\partial \varepsilon}$$

Hence I obtain after applying the chain rule

$$\frac{dI}{d\varepsilon} = \int_{x_a}^{x_b} \left(0 + \frac{\partial F}{\partial Y}\frac{\partial Y}{\partial \varepsilon} + \frac{\partial F}{\partial Y'}\frac{\partial Y'}{\partial \varepsilon}\right)\Bigg|_{\varepsilon=0} dx = 0 \tag{8}$$

I can in fact obtain the expressions $\frac{\partial Y}{\partial \varepsilon}$ and $\frac{\partial Y'}{\partial \varepsilon}$ from equations $Y'(x)$ and $Y(x)$ which will allow me to simplify even further. Partially differentiating equation 7 with respect to ε

$$\frac{\partial Y}{\partial \varepsilon} = \frac{\partial y}{\partial \varepsilon} + \frac{\partial}{\partial \varepsilon}(\varepsilon\eta(x))$$

I know that $y(x)$ is not dependent on ε, so its derivative is 0. This implies

$$\boxed{\frac{\partial Y}{\partial \varepsilon} = \eta(x)} \tag{9}$$

I will differentiate with respect to x the equation 7 to obtain

$$Y'(x) = y'(x) + \varepsilon\eta'(x)$$

I will also partially differentiate it with respect to ε

$$\frac{\partial Y'}{\partial \varepsilon} = \frac{\partial y'}{\partial \varepsilon} + \frac{\partial}{\partial \varepsilon}(\varepsilon\eta'(x))$$

$$\boxed{\frac{\partial Y'}{\partial \varepsilon} = \eta'(x)} \tag{10}$$

Substituting equations 9 and 10 to equation 8 I obtain

$$\boxed{\frac{dI}{d\varepsilon} = \int_{x_a}^{x_b} \left(\frac{\partial F}{\partial Y}\eta(x) + \frac{\partial F}{\partial Y'}\eta'(x)\right)\Bigg|_{\varepsilon=0} dx = 0} \tag{11}$$

The equation can be simplified further to obtain it in the form of Euler-Lagrange. Using integration by parts with respect to x where $u = \frac{\partial F}{\partial Y'}$ and $v' = \eta'(x)$. Then I obtain that

$$u' = \frac{d}{dx}\left(\frac{\partial F}{\partial Y'}\right) \text{ and } v = \eta(x)$$
$$\implies \int_{x_a}^{x_b} \frac{\partial F}{\partial Y'}\eta'(x)dx = \left(\eta(x)\frac{\partial F}{\partial Y'}\right)\bigg|_{x=x_a}^{x_b} - \int_{x_a}^{x_b} \frac{d}{dx}\left(\frac{\partial F}{\partial Y'}\right)\eta(x)dx$$

However I know $\eta(x_a) = 0$ and $\eta(x_b) = 0$ as I have defined, implying that the evaluated expression is 0, therefore

$$\int_{x_a}^{x_b} \frac{\partial F}{\partial Y'}\eta'(x)dx = -\int_{x_a}^{x_b} \frac{d}{dx}\left(\frac{\partial F}{\partial Y'}\right)\eta(x)dx$$

I will substitute this into equation 11 to obtain

$$\frac{dI}{d\varepsilon} = \int_{x_a}^{x_b} \left(\frac{\partial F}{\partial Y}\eta(x) - \frac{d}{dx}\left(\frac{\partial F}{\partial Y'}\right)\eta(x)\right)\bigg|_{\varepsilon=0} dx = 0$$

Factorising $\eta(x)$

$$\frac{dI}{d\varepsilon} = \int_{x_a}^{x_b} \left(\frac{\partial F}{\partial Y} - \frac{d}{dx}\left(\frac{\partial F}{\partial Y'}\right)\right)\eta(x)\bigg|_{\varepsilon=0} dx = 0$$

From the definition of $Y(x)$, that is equation 7, if I evaluate at $\varepsilon = 0$ then $Y(x) = y(x)$ and $Y'(x) = y'(x)$, so my new equation then becomes

$$\frac{dI}{d\varepsilon} = \int_{x_a}^{x_b} \left(\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right)\eta(x)dx = 0$$

For equation 7 to represent a family of curves, I can't let the arbitrary function $\eta(x) = 0$ otherwise variations would also not exist. This implies that the inner bracket must be 0 instead, obtaining me the Euler-Lagrange equation

$$\boxed{\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0} \tag{12}$$

Therefore, Euler-Lagrange allows me to find an extremum function defined as $y(x)$ within the domain of the integral limits. This is done by creating a family of curves resulting from adding some arbitrary function $\eta(x)$ with variational parameter ε to the extremum function $y(x)$, obtaining $Y(x)$. Differentiating the integral with the integrand of the functional $F(x, Y, Y')$ (which now represents a family of curves of functionals) with respect to ε forces this functional to take shape of the extremum function $y(x)$. This implies that the final result I obtain after applying Euler-Lagrange to the surface area of a revolution formula will be the function $y(x)$.

2.5 Beltrami Identity

The Beltrami Identity is a special case of the Euler-Lagrange equation when some functional in the integrand $F(x, y, y')$ is not explicitly dependent on x (by definition this is $\frac{\partial F}{\partial x} = 0$). I can denote such functionals as $F(y, y')$.

Since my surface area a revolution integrand is also not explicitly dependent on x in the integrand, thus I can make use of the identity to find the minimal. I can obtain the Beltrami Identity by applying the Euler-Lagrange equation to a general functional $F(y, y')$ (*Functionals leading to special cases*, n.d.). Hence, I first find $\frac{dF}{dx}$ through the multivariate chain rule:

For y	For y'
$\frac{\partial F}{\partial y}\frac{dy}{dx}$	$\frac{\partial F}{\partial y'}\frac{dy'}{dx}$

$$\implies \frac{dF}{dx} = \frac{\partial F}{\partial y}\frac{dy}{dx} + \frac{\partial F}{\partial y'}\frac{dy'}{dx} \tag{13}$$

I can apply the Euler-Lagrange equation to my functional $F(y, y')$ if I multiply the Euler-Lagrange equation by $\frac{dy}{dx}$ on both sides, therefore

$$\frac{dy}{dx}\frac{\partial F}{\partial y} - \frac{dy}{dx}\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

Now substituting equation 13 to the Euler-Lagrange equation for $\frac{dy}{dx}\frac{\partial F}{\partial y}$ I get

$$\frac{dF}{dx} - \frac{\partial F}{\partial y'}\frac{dy'}{dx} - \frac{dy}{dx}\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

I will factor out the negative to obtain

$$\frac{dF}{dx} - \left(\frac{\partial F}{\partial y'}\frac{dy'}{dx} + \frac{dy}{dx}\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right) = 0 \tag{14}$$

I can re-write one of the inner derivatives as the following

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dy'}{dx}$$

as y is a function which is only dependent on x. Using this information I can write the equation 14 as

$$\frac{dF}{dx} - \left(\frac{\partial F}{\partial y'}\frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{dy}{dx}\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right) = 0 \tag{15}$$

It became evident that the expression inside the bracket is actually the product rule of $\frac{d}{dx}\left(\frac{dy}{dx}\frac{\partial F}{\partial y'}\right)$, implying that we can simply our expression even further. We can confirm by letting $u = \frac{dy}{dx}$ and $v = \frac{\partial F}{\partial y'}$, then

$$u' = \frac{d}{dx}\left(\frac{dy}{dx}\right) \qquad v' = \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)$$

$$\implies \frac{d}{dx}\left(\frac{dy}{dx}\frac{\partial F}{\partial y'}\right) = \frac{\partial F}{\partial y'}\frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{dy}{dx}\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)$$

Hence for equation 15 I can re-express it as

$$\frac{dF}{dx} - \frac{d}{dx}\left(\frac{dy}{dx}\frac{\partial F}{\partial y'}\right) = 0$$

I notice that the expression above is in fact an expression that was differentiated with respect to x all throughout, therefore I can simplify by writing

$$\frac{d}{dx}\left(F - \frac{dy}{dx}\frac{\partial F}{\partial y'}\right) = 0$$

And I can even further simplify if I integrate everything with respect to x

$$\int \frac{d}{dx}\left(F - \frac{dy}{dx}\frac{\partial F}{\partial y'}\right) dx = \int 0 dx$$

Which after integration I obtain that for some arbitrary constant C:

$$F - \frac{dy}{dx}\frac{\partial F}{\partial y'} = C \tag{16}$$

Which is the Beltrami Identity, reduced version of the Euler-Lagrange equation that I can use since my surface area of a revolution integrand is not dependent on x, granting me a simpler way to solve my question.

3 The Problem Of Minimal Surface Area Of A Revolution

3.1 The Extrema Of A Surface Area Of A Revolution

Indeed, due to my problem, I am looking to minimise the surface area of a revolution equation 2, therefore I am looking to apply the Euler-Lagrange equation. Indeed, the surface area of a revolution equation is not dependent on x, hence I could write it as

$$F(y, y') = y(x)\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

Hence I will use the Beltrami identity in order to compute its extrema as planned. I first compute

$$\frac{\partial F}{\partial y'} = y(x)\frac{\partial}{\partial y'}\left(\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\right)$$

$$\implies \frac{\partial F}{\partial y'} = y(x)\frac{\frac{dy}{dx}}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}$$

Applying the Beltrami Identity (equation 16) I obtain

$$y(x)\sqrt{1+\left(\frac{dy}{dx}\right)^2} - y(x)\frac{\left(\frac{dy}{dx}\right)^2}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}} = C$$

Factorising $y(x)$ and multiplying the left expression inside the bracket by $\frac{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}$ to express it as a single fraction

$$y(x)\left(\frac{1+\left(\frac{dy}{dx}\right)^2}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}} - \frac{\left(\frac{dy}{dx}\right)^2}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}\right) = C$$

$$\implies \frac{y(x)}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}} = C$$

In fact, this is a hidden differential equation if I make $\frac{dy}{dx}$ the subject that I have to solve If I want to obtain a clear result of my findings. Raising everything to 2 I get

$$\frac{y^2(x)}{1+\left(\frac{dy}{dx}\right)^2} = C^2$$

And hence rearranging for $\frac{dy}{dx}$

$$y^2(x) = C^2 + \left(C\frac{dy}{dx}\right)^2$$

$$\frac{\sqrt{y^2(x)-C^2}}{C} = \frac{dy}{dx}$$

The $\pm$ is absorbed by the constant C. This is a separable differential equation, therefore

$$\frac{dx}{C} = \frac{dy}{\sqrt{y^2(x)-C^2}}$$

$$\int \frac{dx}{C} = \int \frac{dy}{\sqrt{y^2(x)-C^2}} \tag{17}$$

Whilst the right hand side could be solved with u substitution by letting $y(x) = C\tan(u)$ to obtain an integrand of $\sec(u)$, the solution is non-trivial and complex. Instead, I will use my defined hyperbolic trigonometric identity for a u substitution (equation 3). Let

$$y(x) = C\cosh(u) \tag{18}$$

$$\implies \frac{dy}{du} = C\sinh(u)$$

Substituting in the integral and factoring out C in the denominator

$$\int \frac{C\sinh(u)}{C\sqrt{\cosh^2(u) - 1}} du$$

Using my identity (equation 3) for the denominator and simplifying I obtain

$$\int \frac{C\sinh(u)}{C\sqrt{\sinh^2(u)}} du$$

$$\int du = u$$

Solving for u from my substitution in equation 18

$$\operatorname{arcosh}\left(\frac{y(x)}{C}\right) = u$$

Thus I obtain the solution

$$\int \frac{dy}{\sqrt{y^2(x) - C^2}} = \operatorname{arcosh}\left(\frac{y(x)}{C}\right)$$

Substituting the integral and solving equation 17

$$\frac{x}{C} + h = \operatorname{arcosh}\left(\frac{y(x)}{C}\right)$$

Where h is the constant of integration. Finally rearranging for $y(x)$ I obtain

$$C\cosh\left(\frac{x}{C} + h\right) = y(x)$$

Therefore the above equation is the extrema function $y(x)$ from equation 7 for my particular example of the surface area of a revolution.

3.2 Analysing The Obtained Equation with Soap Film

Well, but what does this really mean? What does the $\cosh(x)$ really tell me for my soap film? Since the $\cosh(x)$ function is even, I can force my equation $C\cosh\left(\frac{x}{C} + h\right)$ to be even too if I let $h = 0$, that is, I will force the symmetry to occur at $x = 0$ to make the investigation and interpretation easier. Then, if I was to restrict the domain with some number α where $\alpha \in \mathbb{R}$ such that the domain is $[-\alpha, \alpha]$, the left side "end" (where $x = -\alpha$) must be identical with the right side "end" (where $x = \alpha$) from $\cosh(\alpha) = \cosh(-\alpha)$, therefore I can assume these "ends" also have identical (same radius) frustums of infinitesimal width after rotating the function 360° along the x axis. The $C\cosh\left(\frac{x}{C}\right)$ function in the domain $[-\alpha, \alpha]$, then, should be hypothetically mimicked by the soap film if I was to sandwich it between two pair of identical real life rings. That is, these rings will force the soap film to take the same properties of $C\cosh\left(\frac{x}{C}\right)$ as these rings will help replicate the fact that $\cosh(\alpha) = \cosh(-\alpha)$. Therefore, in order to investigate this hypothesis, I have cut multiple water pipes of different radius with a saw blade to get approximately the same thickness, and precisely the same radius pairs of rings to investigate. I was able to create a total of 8 rings of 4 distinct radii, which I will denote as r_i where $i \in \{1, 2, 3, 4\}$. Specifically, these radii are $r_1 = 1.35\text{cm}$, $r_2 = 1.10\text{cm}$, $r_3 = 0.85\text{cm}$ and $r_4 = 0.65\text{cm}$. To investigate the equation, however, I have selected r_1 as its radii is the biggest thus the easiest to see. The outcome of the investigation to test the hypothesis can be seen in Figure 4 below, a graph with a photo which I took as I investigated and compared with the function $C\cosh\left(\frac{x}{C}\right)$:

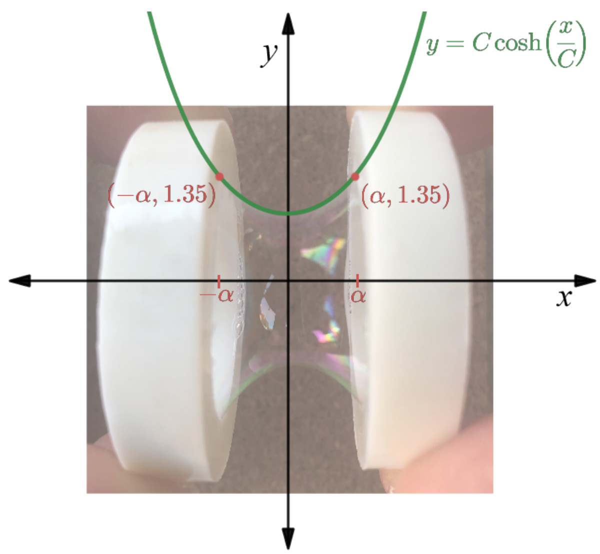

Figure 4: Real life catenoid picture on an axis with function $C\cosh\left(\frac{x}{C}\right)$ (*Desmos Graphing Calculator*, 2011)

The constant C for the image above was found using trial and error, in particular, for the measured distance $\alpha \approx 0.87$, it is $C \approx 0.9$. As hypothesised, the soap film is the rotated function $C\cosh(\frac{x}{C})$ for 360° around the x axis in the interval $[-\alpha, \alpha]$, representing a "catenoid", the 3D rotated variant of the function. However, whilst I have found an expression for a specific value of α, I am looking to apply it to my soap film for all α, as I seek to uncover its mathematical behaviour for all cases of α. To do this algebraically, I have noticed that I can apply my soap film to this equation from the α that I have denoted. I know that for the distance $x = \alpha$ I will always have $y = 1.35$ as seen in figure 4, as the height is y is bounded by the radius of the rings at that point. Hence, for my soap, the following equation holds true

$$\boxed{1.35 = C\cosh\left(\frac{\alpha}{C}\right)} \tag{20}$$

Now reducing the equation to only α and C, I can begin analysing for what values of α and C I can obtain an answer of 1.35 by graphing equation 20 so I can effectively solve for my C for any α value

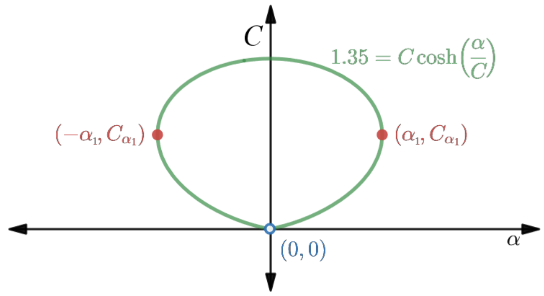

Figure 5: Graph of $1.35 = C\cosh(\frac{\alpha}{C})$ (*Desmos Graphing Calculator*, 2011)

This shows me all the possible value of α with its C values, and the first thing I notice is that we do not have solutions for all $\alpha \in \mathbb{R}$ and I have a maximum attainable α value which I want to find and test out in real life. I first denote that for any r_i its corresponding α maximum coordinates to be (α_i, C_{α_i}). Thus, in the figure above, C_{α_1} was denoted to be the corresponding C value for $\alpha = \alpha_1$ of the equation. Notice that from the definition of the equation, at $C = 0$ the graph becomes discontinuous, as $C \neq 0$. Moreover, the above figure shows me that for $|\alpha| < \alpha_1$ I have 2 solutions, and for $|\alpha| > \alpha_1$ I have no solutions, therefore finding the value of α_1 may be crucial in finding the breaking distance for my soap films, as solutions do not exist for the radius after the distance $|\alpha| > \alpha_1$. To find this value, I will implicitly differentiate with respect to α as the gradient of the tangent is undefined at α_1. Let $u = C$ and $v = \cosh(\frac{\alpha}{C})$, then

$$u' = \frac{dC}{d\alpha} \text{ and } v' = \sinh\left(\frac{\alpha}{C}\right) \cdot \frac{C - \alpha\frac{dC}{d\alpha}}{C^2}$$

Finalising the product rule

$$\sinh\left(\frac{\alpha}{C}\right) \cdot \frac{C - \alpha\frac{dC}{d\alpha}}{C^2} \cdot C + \frac{dC}{d\alpha} \cdot \cosh\left(\frac{\alpha}{C}\right) = 0$$

Re-arranging to make $\frac{dC}{d\alpha}$ the subject

$$-\frac{dC}{d\alpha}\frac{\alpha}{C}\sinh\left(\frac{\alpha}{C}\right) + \frac{dC}{d\alpha}\cosh\left(\frac{\alpha}{C}\right) = -\sinh\left(\frac{\alpha}{C}\right)$$

$$\frac{dC}{d\alpha} = \frac{-\sinh\left(\frac{\alpha}{C}\right)}{-\frac{\alpha}{C}\sinh\left(\frac{\alpha}{C}\right) + \cosh\left(\frac{\alpha}{C}\right)}$$

If I set the denominator 0, I will find all points where the gradients are undefined (or $m \to \infty$ where m is the gradient of the tangent). Hence

$$\boxed{-\alpha\sinh\left(\frac{\alpha}{C}\right) + C\cosh\left(\frac{\alpha}{C}\right) = 0} \tag{21}$$

Well, but what does this mean? This is a very, very interesting result. I know that the catenoid equation must be equal to the radius r_i of the rings. However, radius, which is a constant and the boundary condition of my equation, does not affect the derivative. This means that the derivative of the catenoid equation must apply to **all** catenoids with different radii, that is, it must show the solution to **all** possible maximum distances $x = \alpha$ of different r_i. Consider Figure 6 with different $r_i = C\cosh\left(\frac{\alpha}{C}\right)$ and equation 21 to visualise this relationship:

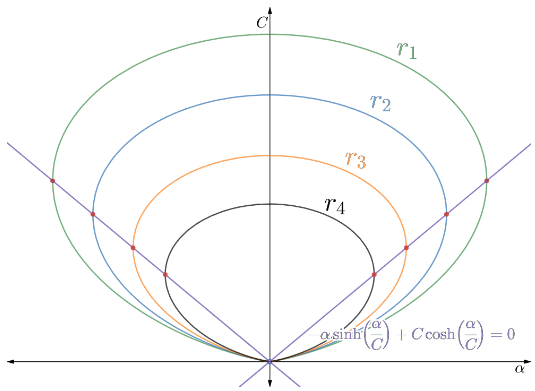

Figure 6: Graphs of $r_i = C\cosh(\frac{\alpha}{C})$ and $-\alpha\sinh(\frac{\alpha}{C}) + C\cosh(\frac{\alpha}{C}) = 0$ (*Desmos Graphing Calculator*, 2011)

Finding the intersections between equation 21 and the catenoid equation is not solvable by hand, as isolating α or C is not possible. Solving for the 2 using technology for all r_i grants me the solutions for α_i in the table below. To test whether the hypothesised α_i breaking distance is the same, I have recorded the breaking distances of my soap film in real life using a high-res camera and slow motion. The recorded distance was denoted to be β_i (By measuring the distance between two rings and dividing by 2 on the breaking frame. See Appendix A for the recordings).

i	r_i	α_i	β_i	$\frac{\lvert\alpha_i-\beta_i\rvert}{\alpha_i}\cdot 100$
1	1.35	0.894704	0.90	0.59%
2	1.10	0.729018	0.75	2.88%
3	0.85	0.563332	0.55	2.37%
4	0.65	0.430783	0.45	4.46%

Table 1: Recorded results of breaking point distance (β_i), hypothesised breaking distance (α_i) and percentage error

The evaluation of these results can be found in section 4. Whilst I was able to find an equation to solve for all α_i and the breaking distance, I have yet to understand why for any distance $|\alpha| < \alpha_i$ I have 2 solutions. Since the Euler-Lagrange does not show the nature of the extrema, it is possible that one solution is a minima, whilst the other is a maxima. I can test out for which solution of C resembles my soap film, as I know that soap film mimics minimal surfaces, which will allow me to negate one of the solutions. To do this, I will consider a arbitrary distance such as $\alpha = 0.5$, and analyse how different C compute the equation $y(x) = C\cosh\left(\frac{x}{C}\right)$(*Introduction to the calculus of variations*, 2016, p.45-46). Using technology, I find that the following are the possible solutions for C when $\alpha = 0.5$:

$$C \simeq 0.187988, 1.24854$$

I graph the function with these C values (graphs of $y = 0.187988\cosh(\frac{x}{0.187988})$ and $y = 1.24854\cosh(\frac{x}{1.24854})$) in the domain of $[-0.5, 0.5]$ as I have chosen from $\alpha = 0.5$, to see how my soap film shape differs with both C

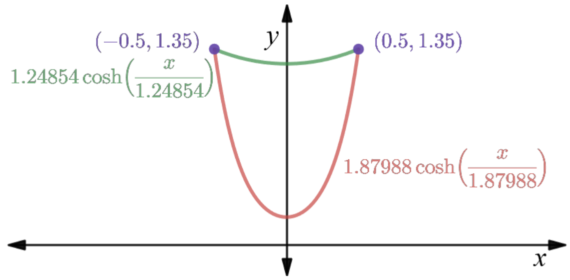

Figure 7: Graph of $y = 0.187988\cosh(\frac{x}{0.187988})$ and $y = 1.24854\cosh(\frac{x}{1.24854})$ in the domain $[-0.5, 0.5]$ (*Desmos Graphing Calculator*, 2011)

It is now possible to see from Figure 7 that for larger C, the function arc length is smaller than the other C solution function. Moreover, the shape with $C = 1.24854$ largely corresponds with the shape in my photo if it is rotated 360° around the x axis! Thus, I can deduce that for values of $C > C_{\alpha_i}$ it is a minimal.

4 Evaluation and Conclusion

The function that I've found to express soap films between rings and find the breaking distance shown in table 1 seems to be accurate through real life testing! However, deviations in results in the table can occur from factors such as thickness of the rings that I have used, the surrounding air pressure or even the concentration of the soap solution. For a more accurate model in computers, these would also have to be mathematically defined. Moreover, given that I've used a ruler to both measure the radius and the distance between the rings, I was prone to parallax error, as well as limited accuracy of measurement to 1 decimal point. Despite this, however, my equation to solve for α_i for any r_i seems to be a rather good approximation for the breaking distance β_i in room conditions. This means that I can use equation 20 whilst restricting the value of my C to be $C \geq C_{\alpha_i}$ for some distance α to then compute C and thus find a function that would represent the soap film for that particular value of α, and finally rotate that function 360° around the x axis to obtain the final simulation.

However, equation 21 also caught my eye as it seems that its relationship is almost linear as seen in figure 6, therefore it is worth considering and checking whether all α_i of their respective r_i follow some linear equation and thus a ratio. This can be useful in the future for optimisation of my model, as the computer will simply have to downscale and upscale the breaking distance for some constant, without having to compute each solution for catenoid using the equation of the derivative which can take significantly more processing power.

The main tool of this investigation which I had to learn and use, the Euler-Lagrange equation, undoubtedly brings many opportunities for me to further define even more mathematical relationships for modeling. One of such mathematical relationships is the newly introduced Ray Tracing technology in graphics. This technology is defined through Snell's law, which is also derived from the Euler-Lagrange equation and Fermat's principle (light always seeks minimal path).

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Appendices

A Recordings of Breaking Distances α_i

For the availability, privacy, and security of these videos, the recordings were uploaded and are available as unlisted.

These were uploaded using an anonymous YouTube account.

Recording using r_1 rings for α_1

https://youtu.be/u1CGKfzeXLw

Recording using r_2 rings for α_2

https://youtu.be/A8n7OSjcINY

Recording using r_3 rings for α_3

https://youtu.be/gj7ul-aR6Rg

Recording using r_4 rings for α_4

https://youtu.be/4wBY77QKvWs